View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Memorial University Research Repository

Some near horizon studies on Weyl and Vaidya spacetimes

by

©Sharmila Dhevi G G

A Thesis submitted to the School of Graduate Studies in partial fulfillment of the requirements for the degree of

Master of Science in Mathematics

Department of Mathematics and Statistics

October 2015

Memorial University of Newfoundland

St John's, Newfoundland

Abstract

This thesis comprises some studies on the Weyl, Vaidya and Weyl distorted Schwarzschild (WDS) spacetimes. The main focal areas are : a) construction of near horizon metric(NHM) for WDS spacetime and subsequently a "stretched horizon" prescribed by the membrane formalism for black holes, b) application of membrane formalism and construction of stretched horizons for Vaidya spacetime and c) using the thin shell formalism to construct an asymptotically flat spacetime with a Weyl interior where the construction does not violate energy conditions. For a), a standard formalism developed in [1] has been used wherein the metric is expanded as a Taylor series in ingoing Gaussian null coordinates with the affine parameter as the expansion parameter. This expansion is used to construct a timelike "stretched horizon" just outside the true horizon to facilitate some membrane formalism studies, the theory for which was first introduced in [2]. b) applies the membrane formalism to Vaidya spacetime and also extends a part of the work done in [1] in which event horizon candidates were located perturbatively. Here, we locate stretched horizons in close proximity to every event horizon candidate located in [1]. c) is an attempt to induce Weyl distortions with a thin shell of matter in an asymptotically flat spacetime without violating energy conditions.

Key Words : weyl distortion, weyl spacetimes, near horizon metric, membrane formalism, stretched horizon, thin shell formalism, energy conditions, vaidya spacetime

Acknowledgements

I am indebted to my supervisor, Ivan Booth without whose support, guidance, ideas and encouragement, this thesis and also the entire course of my masters study would not have been possible. I am very grateful for his support in many ways during conferences and journal club meetings, which has given me a broader view of research in mathematical relativity. I thank Ivan Booth and Hari Kunduri for all the enriching discussions on a wide range of topics in mathematics and physics. I thank all my instructors here :- Eduardo-Martinez Pedroza, Deping Ye, Hari Kunduri, Marco Merkli, Jie Xiao and Yuri Bahturin for their wonderful courses and teaching which has made my transition from physics to mathematics quite a delightful experience. I thank my family for their unquestioning support, unconditional love and most of all, for their every effort to make me feel closer to home despite the distance. I am very grateful to Robb Mann for a wonderful research experience and all the advice that decided the course of my graduate studies. I thank the administrative staff for easing my official formalities many a time. I thank the mathematics department at Memorial for a great research environment. I am thankful to David Tian and Aghil Alaee for helpful conversations during my initial days here. My heartfelt thanks goes to all my friends, in particular to JP and everyone in St. John's. Finally, I thank the reviewers in advance for their helpful comments.

Table of Contents

List of Tables

List of Figures

Notations and Symbols

The metric signature is chosen to be (-,+,+,+). We use greek letters $(\alpha, \beta, \mu, \nu..)$ to denote indices on the full spacetime, italicized alphabets $(a, b, c..)$ for indices on a codimension one hypersurface and $(A, B, C..)$ to denote indices on a codimension two hypersurface. χ is used to represent coordinate patches. Σ and S are used to denote codimension one and codimension two hypersurfaces respectively and $d\Sigma^2$ (and h_{ab}) and dS^2 (and q_{AB}), the corresponding induced metrics on the respective hypersurfaces. K_{ab} stands for extrinsic curvature and $k^{(n)}_{AB}$ and $k^{(\ell)}_{AB}$ represent the analogues of extrinsic curvature for codimension two hypersurfaces. $\triangleq|_{\mathcal{F}}$ is a relation symbol used to denote that an expression or a quantity on the right hand side of the symbol is evaluated on or at $\mathcal{F}$, where $\mathcal{F}$ is a hypersurface. All other symbols and notations have their usual meanings.

Chapter 1

Introduction

One of the striking features of general relativity(GR) ([3], [4] and [5]) is the existence of black hole spacetimes. Research on black holes has captivated mathematicians and physicists alike and perhaps the most common way of studying black holes is through the exact solutions of Einstein equations.

Most of the work in this thesis is based on the Weyl distorted Schwarzschild(WDS) ([6] and [7]) and Vaidya [8] spacetimes. WDS is a distorted version of the Schwarzschild spacetime which nevertheless still contains a singularity like its Schwarzschild predecessor [9]. This thesis is partly devoted to the study of the near horizon region in this spacetime and to developing a better understanding of Weyl distortions. We have also worked on slowly evolving horizons(SEH) in Vaidya spacetime ([10] and [11]). In what follows, we hope to motivate the questions that we have pursued in these topics. To start with, we discuss the Weyl, Schwarzschild and Vaidya spacetimes to highlight some of their features.

1.1 Weyl, Schwarzschild and Vaidya spacetimes

The black hole spacetimes that are studied usually have some underlying symmetries. There are motivations and reasons for studying spacetimes with symmetries. Mathematically, this means that the manifold ($\mathcal{M}$) in question is unaltered by the respective symmetry group action. Hence, all studies concerning the spacetime $(\mathcal{M}, g)$ can be brought down to the quotient space or orbit space of the manifold under the action of this symmetry group. It is important to note that this can be done without any loss of information concerning spacetime dynamics. Physically, symmetries are useful in capturing some defining features of the spacetime like being static, stationary or axisymmetric. Here we discuss three such spacetimes with symmetries - Schwarzschild, Weyl and Vaidya.

1.1.1 General Weyl solutions

The Weyl metric is the most generic static and axisymmetric solution to the vacuum Einstein field equations [7]. So, every vacuum static axisymmetric spacetime can be written in the Weyl form in a unique way. The Weyl metric reads

$$ds^2 = -\mathrm{e}^{2A} dt^2 + \mathrm{e}^{2B-2A}(d\rho^2 + dz^2) + \mathrm{e}^{-2A} d\phi^2. \tag{1.1}$$

Eq.(1.1) has been written out in cylindrical coordinates. Here, $A = A(\rho, z)$ and $B = B(\rho, z)$ are axisymmetric functions called Weyl potentials. The spacetime is static as it admits a timelike Killing vector field, viz., ∂_t which is orthogonal to spacelike hypersurfaces. It also admits ∂_ϕ as a Killing vector field which generates a one parameter isometry group with periodic orbits. This makes the spacetime axisymmetric. The vacuum field equations without cosmological constant are given by

$$Ric_{\mu\nu} = 0. \tag{1.2}$$

The form of Eq.(1.1) does not guarantee that the Einstein equations are satisfied. For Eq.(1.1) to be a solution of the Einstein equations, A and B must satisfy

$$\frac{\partial^2 A}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial A}{\partial \rho} + \frac{\partial^2 A}{\partial z^2} = 0, \tag{1.3}$$

$$\frac{\partial B}{\partial \rho} = \rho\left\{\left(\frac{\partial A}{\partial \rho}\right)^2 - \left(\frac{\partial A}{\partial z}\right)^2\right\} \quad \text{and} \tag{1.4}$$

$$\frac{\partial B}{\partial z} = 2\rho\frac{\partial A}{\partial \rho}\frac{\partial A}{\partial z}. \tag{1.5}$$

Eq.(1.3) is the Laplace equation in three dimensions, which is homogeneous and linear. So, two different solutions of Eq.(1.3) can be superposed to get a new solution. As it can be seen from Eqs.(1.4) and (1.5), the same is not true for B. Say we have two solutions of the Laplace equation, A_1 and A_2, so that

$$\frac{\partial^2 A_1}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial A_1}{\partial \rho} + \frac{\partial^2 A_1}{\partial z^2} = 0 \quad \text{and} \tag{1.6}$$

$$\frac{\partial^2 A_2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial A_2}{\partial \rho} + \frac{\partial^2 A_2}{\partial z^2} = 0. \tag{1.7}$$

Then, for $\tilde{A} = A_1 + A_2$, Eqs. (1.6) and (1.7) are are sufficient to imply that

$$\frac{\partial^2 \tilde{A}}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \tilde{A}}{\partial \rho} + \frac{\partial^2 \tilde{A}}{\partial z^2} = 0. \tag{1.8}$$

If B_1 and B_2 are auxiliary potentials defined by A_1 and A_2 respectively, then,

$$\frac{\partial B_1}{\partial \rho} = \rho\left\{\left(\frac{\partial A_1}{\partial \rho}\right)^2 - \left(\frac{\partial A_1}{\partial z}\right)^2\right\}, \tag{1.9}$$

$$\frac{\partial B_1}{\partial z} = 2\rho\frac{\partial A_1}{\partial \rho}\frac{\partial A_1}{\partial z}, \tag{1.10}$$

$$\frac{\partial B_2}{\partial \rho} = \rho\left\{\left(\frac{\partial A_2}{\partial \rho}\right)^2 - \left(\frac{\partial A_2}{\partial z}\right)^2\right\} \quad \text{and} \tag{1.11}$$

$$\frac{\partial B_2}{\partial z} = 2\rho\frac{\partial A_2}{\partial \rho}\frac{\partial A_2}{\partial z}. \tag{1.12}$$

Now, equations for $\tilde{B}$ ($\tilde{B}$ is auxiliary to $\tilde{A}$) in terms of A_1 and A_2 are

$$\frac{\partial \tilde{B}}{\partial \rho} = \frac{\partial B_1}{\partial \rho} + \frac{\partial B_2}{\partial \rho} + 2\rho\left\{\frac{\partial A_1}{\partial \rho}\frac{\partial A_2}{\partial \rho} - \frac{\partial A_1}{\partial z}\frac{\partial A_2}{\partial z}\right\} \quad \text{and} \tag{1.13}$$

$$\frac{\partial \tilde{B}}{\partial z} = \frac{\partial B_1}{\partial z} + \frac{\partial B_2}{\partial z} + 2\rho\left\{\frac{\partial A_1}{\partial \rho}\frac{\partial A_2}{\partial z} + \frac{\partial A_2}{\partial \rho}\frac{\partial A_1}{\partial z}\right\}. \tag{1.14}$$

This demonstrates non-linearity. Further, Eqs.(1.8), (1.13) and (1.14) are particularly useful when working with Weyl distortions of a particular spacetime. A similar approach was used in [13].

Since we work with WDS, a suitable coordinate system to use would be the one adapted to the Schwarzschild metric (discussed in the next subsection) i.e. radial/spherical coordinates. Eq.(1.1) written out in spherical coordinates reads

$$ds^2 = -e^{2A}dt^2 + e^{-2A+2B}\left(dr^2 + r^2 d\theta^2\right) + e^{-2A}r^2\sin^2\theta d\phi^2. \tag{1.15}$$

The field equations become

$$\frac{\partial^2 A}{\partial r^2} + \frac{2}{r}\frac{\partial A}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial A}{\partial \theta} = 0, \tag{1.16}$$

$$\frac{\partial B}{\partial r} = 2\,\sin\theta\cos\theta\frac{\partial A}{\partial r}\frac{\partial A}{\partial \theta} + r\sin^2\theta\left(\frac{\partial A}{\partial r}\right)^2 - \frac{\sin^2\theta}{r}\left(\frac{\partial A}{\partial \theta}\right)^2 \text{ and} \tag{1.17}$$

$$\frac{\partial B}{\partial \theta} = -r^2\cos\theta\sin\theta\left(\frac{\partial A}{\partial r}\right)^2 + \cos\theta\sin\theta\left(\frac{\partial A}{\partial \theta}\right)^2 + 2r\sin^2\theta\frac{\partial A}{\partial r}\frac{\partial A}{\partial \theta}. \tag{1.18}$$

Note that the spacetime described by Eq.(1.1) or equivalently by Eq.(1.15) is not generically regular along the symmetry axis [15]. This means that conical singularities may be present (see Appendix A for details). To see this, we take two small circles about the axis of rotation (i.e. vanishing ∂_ϕ) at $\theta = 0$ and $\theta = \pi$. Let the range of ϕ be $[0, 2\pi D)$. D is included here to enable some flexibility in rescaling the angular coordinate. Ideally, the circumference to radius ratios for the two circles should match. In case this ratio is not equal to 2π, we can adjust D so that it is finally 2π. Examining Eq.(1.15), for a circle about $\theta = 0$, we have

$$\frac{\text{Circumference}}{\text{Radius}} = \lim_{\theta \to 0} \frac{e^{-B} 2\pi D \sin\theta}{\theta} = 2\pi D e^{-B(0)}. \tag{1.19}$$

For a circle about $\theta = \pi$, we have

$$\frac{\text{Circumference}}{\text{Radius}} = \lim_{\theta \to \pi} \frac{e^{-B} 2\pi D \sin(\pi - \theta)}{\pi - \theta} = 2\pi D e^{-B(\pi)}. \tag{1.20}$$

So, D cannot be consistently fixed unless $B(0) = B(\pi)$ or if $B(\theta)$ vanishes along the axis. Physically $B(0) \neq B(\pi)$ represents a conical singularity. Geometrically the spacetime structure would either have an angle deficit (called cosmic string) or an angle excess (called cosmic strut) [14]. This analysis similarly holds for WDS mentioned onward. Hence, WDS might also contain conical singularities.

1.1.2 Schwarzschild metric

The Schwarzschild metric describes the spacetime due to a spherical massive object. The metric reads

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{1.21}$$

Here M is interpreted as mass of the body. $r = 2M$ is the event horizon for the spacetime. The spacetime is static with ∂_t, ∂_ϕ, $\sin\phi\,\partial_\theta + \cot\theta\cos\phi\,\partial_\phi$ and $\cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi$ as Killing vector fields. A Killing horizon is located at $r = 2M$, which corresponds to the vanishing of the norm of ∂_t. Since the spacetime is spherically symmetric, it admits an SO(3) action. The Schwarzschild spacetime is asymptotically flat and it has been proved that any asymptotically flat spherically symmetric static solution of the vacuum Einstein equations should have a Schwarzschild exterior - this goes by the name of Birkhoff's theorem [5].

It was mentioned in the preceding section that every vacuum static axisymmetric solution can be written in the Weyl form in a unique way. Below, we validate this statement for the case of Schwarzschild ([6] and [14]). With the following coordinate transformation (called prolate spheroidal coordinates),

$$r = M(x+1), \quad \cos\theta = y, \tag{1.22}$$

Eq.(1.21) becomes

$$\begin{aligned} ds^2 = &-\left(\frac{x-1}{x+1}\right)dt^2 + M^2\left(\frac{x+1}{x-1}\right)dx^2 + M^2\frac{(x+1)^2}{(1-y^2)}dy^2 \\ &+ M^2\,(x+1)^2\left(1-y^2\right)d\phi^2. \end{aligned} \tag{1.23}$$

Since the metric should still be Lorentzian, $|y| < 1$ always. With another coordinate transformation (illustrated in Fig.[1.1]),

$$\rho = M\sqrt{(x^2-1)\,(1-y^2)} \quad , \quad z = Mxy, \tag{1.24}$$

Eq.(1.21) takes the Weyl form in Eq.(1.1) for which the potentials are given by

$$A = \frac{1}{2}\log\left\{\frac{U_- + z + M}{U_+ + z + M}\right\} \quad \text{and} \quad B = \frac{1}{2}\log\left\{\frac{(U_+ + U_-)^2 - 4M^2}{4U_+U_-}\right\}. \tag{1.25}$$

Here $U_\pm^2 = \rho^2 + (z \pm M)^2$. Classically, Eq.(1.25) is interpreted as the potential for a finite rod present along $\rho = 0$ with a mass per unit length of 1/2, length $2M$ and total mass M.

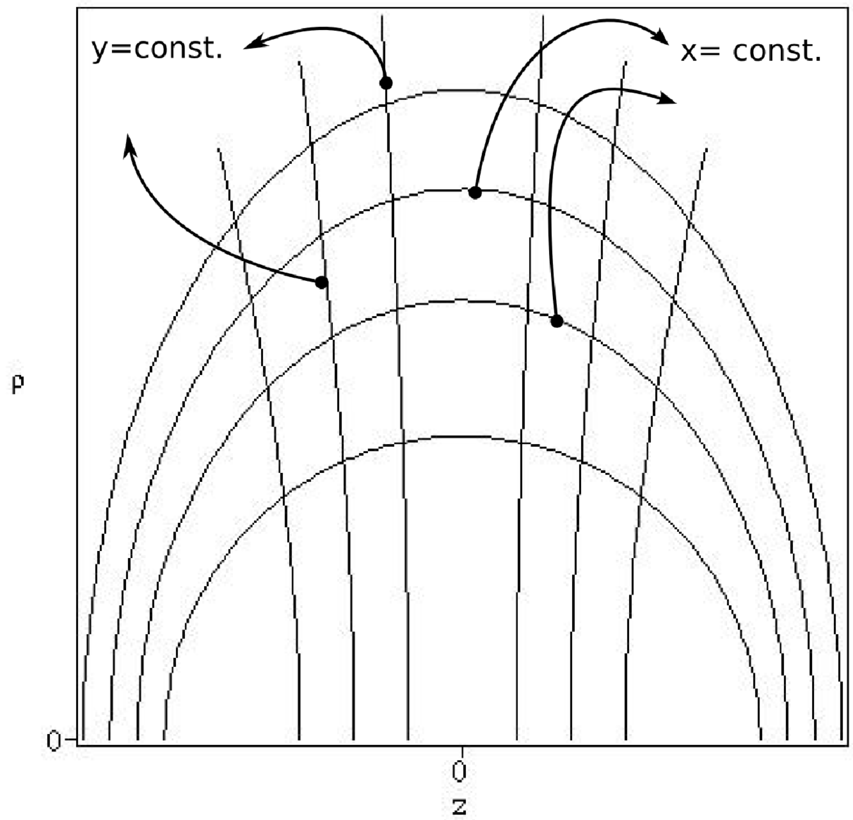


Figure 1.1: Illustration of coordinate transformation in Eq.(1.24)

Figure based on [14].

It should be noted that Weyl spacetimes are not necessarily asymptotically flat. So, defining global energy and other quantities is ambiguous and difficult. This has been highlighted in a recent work [16] for the more complex case of Melvin-Kerr-Newmann spacetime. Also, the black hole uniqueness theorems do not apply here.

1.1.3 Vaidya spacetime

The Vaidya metric describes the spherically symmetric spacetime of a star or black hole absorbing or emitting null dust ([4] and [8]). The Vaidya metric can be obtained by "radiating" the Schwarzschild metric as follows. The Schwarzschild metric given by Eq.(1.21) written in ingoing null coordinates (v, r, θ, ϕ) defined as

$$t = v - r - 2M \ln\left(\frac{r}{2M} - 1\right) \qquad \text{or} \qquad dt = dv - \frac{dr}{\left(1 - \frac{2M}{r}\right)} \tag{1.26}$$

becomes

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{1.27}$$

$M = M(v)$ gives the Vaidya metric written below, which is still physically meaningful.

$$ds^2 = -\left(1 - \frac{2M(v)}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{1.28}$$

Eq.(1.28) represents a black hole that is being irradiated by infalling null dust. The stress energy tensor is given by

$$T_{ab} = \frac{dM/dv}{4\pi r^2}[dv]_a[dv]_b. \tag{1.29}$$

To identify horizons in this spacetime, we first construct outward and inward oriented null vectors ℓ^a and n^a,

$$\ell^a = \frac{\partial}{\partial v} + \frac{1}{2}\left(1 - \frac{2M}{r}\right)\frac{\partial}{\partial r} \quad \text{and} \tag{1.30}$$

$$n^a = -\frac{\partial}{\partial r}. \tag{1.31}$$

The expansion of a hypersurface in the direction X^a is expressed as

$$\theta_{(X)} = \tilde{q}^{AB} k^{(X)}_{AB} \quad \text{where} \tag{1.32}$$

$$k^{(X)}_{AB} = e^a_A e^b_B \nabla_a X_b. \tag{1.33}$$

Here e^a_A are projections and $\tilde{q}_{AB}$ is the induced metric on the hypersurface.

Expansions along ℓ^a and n^a are :

$$\theta_{(\ell)} = \frac{r-2M}{r^2} \quad \text{and} \tag{1.34}$$

$$\theta_{(n)} = -\frac{2}{r}. \tag{1.35}$$

We have a dynamical horizon at $r = 2M(v)$ which corresponds to $\theta_{(\ell)} = 0$ (see Appendix A , [4] and [17] for details). The symmetries of this spacetime are three Killing vector fields associated with spherical symmetry. This is not a vacuum solution and it is also not static. Hence it does not fall into the class of Weyl metrics. It is important to note that the Vaidya spacetime does not contain a Killing horizon which makes the definition of surface gravity difficult.

1.2 Weyl distorted Schwarzschild(WDS) spacetime

WDS can be thought of as a distorted version of the Schwarzschild spacetime. In spherical coordinates, the metric reads

$$\begin{aligned} ds^2 = &- e^{2A}\left(1-\frac{2M}{r}\right)dt^2 + e^{-2A+2B}\left\{\frac{dr^2}{\left(1-\frac{2M}{r}\right)} + r^2 d\theta^2\right\} \\ &+ e^{-2A} r^2 \sin^2\theta d\phi^2. \end{aligned} \tag{1.36}$$

where $A = A(r,\theta)$ and $B = B(r,\theta)$ are axisymmetric potentials. The Einstein field equations for A and B are

$$\frac{\partial^2 A}{\partial r^2} = \frac{1}{(r-2M)\,r}\left\{2M\frac{\partial A}{\partial r} - \left(\frac{\partial A}{\partial \theta}\right)\cot\theta - 2\left(\frac{\partial A}{\partial r}\right)r - \left(\frac{\partial^2 A}{\partial \theta^2}\right)\right\}, \tag{1.37}$$

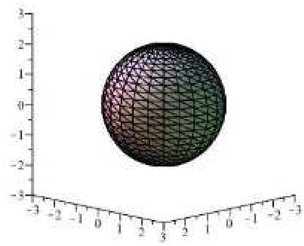
$$\frac{\partial B}{\partial r} = \frac{\sin\theta}{r^2 + M^2\sin^2\theta - 2Mr}\left\{\begin{array}{c}(M-r)\left(\frac{\partial A}{\partial\theta}\right)^2 + 2M\sin\theta(r-M)\left(\frac{\partial A}{\partial\theta}\right)+ \\ r\sin\theta\left(\frac{\partial A}{\partial r}\right)^2(r-2M)(r-M) \\ +2M\cos\theta\left(\frac{\partial A}{\partial\theta}\right) + 2r\cos\theta\left(\frac{\partial A}{\partial\theta}\right)\left(\frac{\partial A}{\partial r}\right)\end{array}\right\} \text{ and} \tag{1.38}$$

$$\frac{\partial B}{\partial\theta} = \frac{\sin^2\theta}{r^2 + M^2\sin^2\theta - 2Mr}\left\{\begin{array}{c}r^2\cot\theta\,(r-2M)^2\left(\frac{\partial A}{\partial r}\right)^2 - 2Mr\cot\theta\left(\frac{\partial A}{\partial\theta}\right)^2 \\ -2Mr\,(r-2M)\cot\theta\left(\frac{\partial A}{\partial r}\right) \\ +\left(\frac{\partial A}{\partial\theta}\right)(2Mr + r^2\cot\theta - 2M^2) \\ 2r\,(r-M)\,(r-2M)\left(\frac{\partial A}{\partial r}\right)\left(\frac{\partial A}{\partial\theta}\right)\end{array}\right\}. \tag{1.39}$$

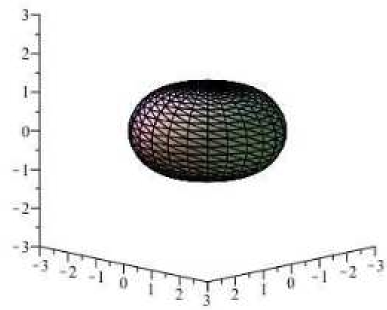
These distortions change some important Schwarzschild features - WDS is neither spherically symmetric nor asymptotically flat. WDS also inherits the possibility of a conical singularity from the Weyl solution. But, throughout this thesis, we will restrict our attention to spacetimes which do not contain any conical singularities. Despite these differences with the Schwarzschild spacetime, some key black hole properties remain in WDS. The spacetime is still static, axisymmetric and there is an isolated horizon at $r = 2M$ which is also a Killing horizon ([7] and [9]).

Previous studies on WDS [9] have shown how the Schwarzschild horizon is distorted by Weyl potentials, wherein it was also confirmed that WDS contains trapped surfaces and a singularity. Some of the horizon distortions illustrated in [9] are shown below in Fig.[1.2]. The figures show the distorted horizon for the special case of quadrupolar distortions (akin to multipole moments in electromagnetism). Quadrupolar distortion means that the potential $A(r, \theta)$ is expanded in terms of Legendre polynomials and the only non-zero coefficient in the expansion is the second one. $B(r, \theta)$ has been calculated using the Einstein equations mentioned earlier i.e., Eqs.(1.38) and (1.39) .

Gaining a better understanding of some near horizon aspects is the main motivation for our study of WDS and Vaidya spacetimes. In the next section, we discuss in some detail, our reasons for studying WDS and the approaches we have used so far to understand horizon distortions. The study with Vaidya spacetime is aimed at gaining an understanding of the near equilibrium regime, more specifically the slow evolution of geometry and fields in nearly isolated horizons. The aspects of Vaidya spacetime that we have looked into is also briefly discussed in the following section.

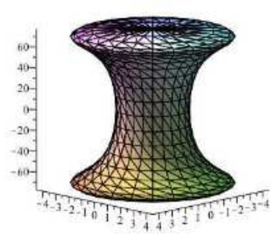


(a) Schwarzschild horizon



(b) Small positive quadrupolar distortion

(c) Small negative quadrupolar distortion



(d) Large negative quadrupolar distortion

Figure 1.2: Horizon distortions in WDS

Figures from [9]

1.3 Motivation

In the case of Weyl distortions, for all the approaches that we have and might come up with in future, we would like our end goal to lead to a better physical understanding behind the generation of Weyl distortions. With our studies in Vaidya spacetime, we would like know if the notion of "stretched horizon" prescribed by the membrane formalism can be fused with the known quasilocal classifications of horizons in the near equilibrium or nearly isolated regime of horizons. The membrane formalism, thin shell formalism and the topics of study in Vaidya spaetime are briefly introduced here with more details in subsequent chapters.

1.3.1 Membrane formalism

The membrane formalism was introduced in [2] to simplify astrophysical studies where the full mathematical theory of black holes was unnecessary, even though the effects of the horizon would need to be taken into account. Within this formalism, all the effects of the horizon relevant to the external observer are taken care of by the attributes of the "stretched horizon" which is a timelike surface just outside the true horizon. The membrane formalism contains beautiful laws that can be intuitively understood as laws for the stretched horizon, which give it a membrane-like fluid structure with viscosity, electrical conductivity and temperature - to name a few attributes. This provides an effective intuitive picture for astrophysics, albeit a fictitious one. The laws that hold on the horizon also work well to a good approximation on the stretched horizon. We need to work with the near horizon spacetime since the stretched horizon is and should be very close to the true horizon to capture its effects on the external universe. The fact that this surface is timelike rather than null makes the stretched horizon relevant to the external observer. To understand Weyl distortions better, we have worked out how the Weyl potentials distort the Schwarzschild stretched horizon.

1.3.2 Near horizon studies with Vaidya spacetime

We would like to know how, in the near equilibrium regime, the stretched horizon compares with the other known notions of quasilocal horizons. The notion of a slowly evolving horizon (SEH) has been discussed in [11]. We answer this question for the Vaidya spacetime before addressing it in full generality. The Vaidya spacetime is a reasonable choice because a) it is not stationary, which allows a meaningful study of the near equilibrium limit and b) it is the simplest example of a dynamical horizon because of spherical symmetry, which would make the near horizon calculations easier. This work is an indication that one can define a broader class of horizons which could include the event horizon, slowly evolving horizon and stretched horizon.

1.3.3 Thin shell formalism

The thin shell formalism was introduced by Israel [18] to consistently paste two different spacetimes along a hypersurface where the induced metric is the same from both the contributing spacetimes. Any discontinuities that arise at the level of connections is interpreted as a thin layer of matter on the hypersurface.

The Laplace equation in three dimensions forms a part of the Einstein equations for the Weyl metric i.e., Eq.(1.16). The solution space to the Laplace equation is a direct sum of two subspaces which are - a) solutions which are singular at the origin and b) solutions which are singular at infinity. In our work, horizon distortions are sourced by Weyl potentials. Hence, it is natural to eliminate the case of sources coming from the origin which refers to a) because it would mean that distortions are somehow caused by the spacetime singularity. So, we try to study how distortions caused by b) can be simulated in an asymptotically flat spacetime. One known approach which may solve this problem is the thin shell formalism. This is work in progress. Through the thin shell formalism, we hope to obtain a realistic stress energy tensor (obeying certain energy conditions discussed in Chapter 3) that could be causing such a distortion in an otherwise asymptotically flat spacetime. Achieving this construction could provide scope for a matter model that generates Weyl distortions.

1.4 Outline

We provide a synopsis of subsequent chapters in this section.

Chapter 2 discusses the membrane formalism approach to WDS. Here we have described the method used for obtaining the near horizon metric(NHM). This has been obtained in [1] by computing deformations by an evolution vector field. The chapter starts with a review of the initial value formulation in GR. After constructing the NHM, we use it to apply the membrane formalism to WDS. The result shows how Weyl potentials distort the stretched horizon for Schwarzschild spacetime. A comparison is also made with the stretched horizon of Schwarzschild spacetime.

Chapter 3 applies the membrane formalism to Vaidya spacetime in the slowly evolving regime which makes it easier to approximate a definition for surface gravity. In this chapter, we locate stretched horizons near event horizon candidates which were located using perturbative solutions in [1]. We conclude that in the near equilibrium regime, one can group the known quasilocal notions of horizons which includes the slowly evolving horizon and stretched horizon to a larger category for the case of Vaidya spacetime.

Chapter 4 will describe the thin-shell formalism in some detail and outline the attempts to construct an asymptotically flat spacetime with a Weyl interior using this formalism. This is done to provide asymptotic flatness and induce the Weyl distortions with a thin shell of matter. It will be seen that this embedding is not quite straightforward. In this section, we hope to capture the effects Weyl distortions which have singularities at infinity in a physically meaningful surface stress energy tensor. Some energy conditions needed to ensure this meaningfulness are also discussed in this chapter.

In Chapter 5, we will summarize our work so far outlining the results of membrane formalism for WDS, near horizon studies in the slowly evolving regime of Vaidya spacetime and thin shell formalism for Weyl distortions. We also mention the current directions of research that we are pursuing - some laws for a broader class of slowly evolving horizons in a general setting and more attempts on embedding Weyl distortions in asymptotically flat spacetimes.

Chapter 2

Near horizon metric(NHM) and stretched horizon for WDS

In the literature ([19] [20] [21] [22] and [23]), "NHM" often has been used to refer to a different approach from what has been adopted here. The NHM in those cases is defined exclusively for black holes with degenerate Killing horizons like extremal holes. But here, "NHM" is used to refer to the Taylor series expansion of a metric "about the horizon" and we use the formalism developed in [1] to do this.

We would like to construct spacetime near the isolated horizon of a black hole with the only information available being data on the horizon. This data includes not just the first and second derivatives of the metric (which is data in the context of initial value formulation of GR), but derivatives up to all orders. This is used to construct the NHM assuming the convergence of the corresponding Taylor series. An immediate question that comes to mind is - "How does this compare with the initial value formulation of GR ?". This chapter starts with a review of initial value formulation to answer this question. We will see how finding the NHM can be addressed from the initial data point of view. It turns out in our case that the initial value problem is not well posed with standard initial data. Hence we use the formalism developed in [1] for the construction. In this formalism, the problem is addressed like the standard initial value problem to begin with. The horizon which is a null 3-surface is then broken down into a union of spacelike 2-surfaces(foliations). Using analogues for extrinsic curvature and constraint equations (a.k.a Gauss-Codazzi equations similar to the initial value formulation), evolution equations are obtained by studying the deformation caused by a vector field. Since the standard initial data is not sufficient, the NHM constructed this way is dependent on more information from the horizon after the sub-leading order in the expansion. The final expression of NHM is a Taylor series expansion in ingoing Gaussian null coordinates with the ingoing radial affine parameter as the expansion parameter. We end this chapter with an illustration of this formalism including explicit calculations for the stretched horizon in the case of WDS.

2.1 Initial value problem(IVP) of GR

The initial value problem(or formulation) of GR also goes by the name of 3+1 formalism for GR or Cauchy problem in GR ([3], [5], [26], [28] and [29]). Similar to the theory of differential equations, one tries to find a solution and determine the dynamics and global behavior of the solution using initial data and the differential equation in hand. The Einstein field equations, Eq.(1.2) are quasi-linear second order partial differential equations for the components of $g_{\mu\nu}$. The initial value formulation looks at these equations from an evolutionary point of view. Because of the highly non-linear nature of Einstein field equations, well-posedness isn't always guaranteed and this is also the case for WDS. In fact, the question of how meaningful terms like 'Cauchy problem' and 'initial data' are for GR was fully conceptually resolved only in 1969 [30]. Studying the Cauchy problem is quite central to studying singularities. Essentially, in the IVP setting, one considers the possibilty of $\mathcal{M} = \Sigma\times\mathbb{R}$. The family $\{\Sigma_t\}_{t\in\mathbb{R}}$ is called slicings or foliations as shown in Fig.[2.1] below.

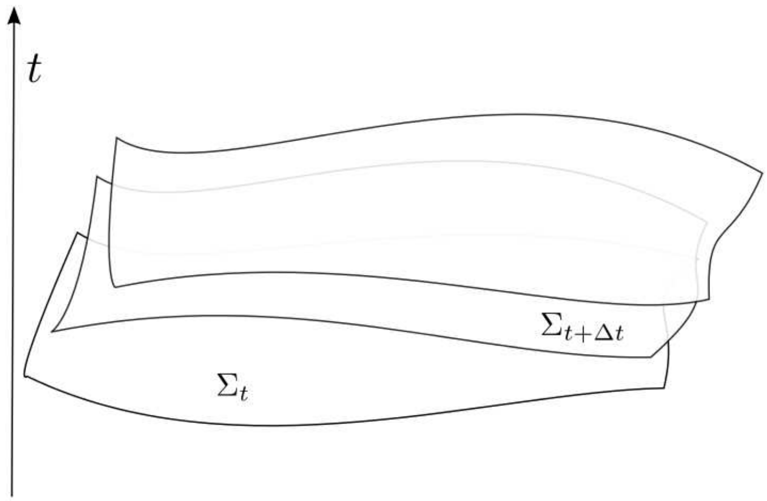


Figure 2.1: Spacetime foliations

It happens to be that, for this split to be reasonable, Σ needs to be a smooth closed achronal hypersurface called a Cauchy surface. The existence of such a surface is necessary and sufficient to ensure that one can meaningfully talk about dynamics (this property of the spacetime is called global hyperbolicity). The information on a Cauchy surface constitutes initial data for the field equations. Initial data is comprised of the induced metric (h_{ij}) and extrinsic curvature (K_{ij}) on the Cauchy surface. K_{ij} can be thought of as a derivative of the induced metric. K_{ij} measures how the hypersurface curves in the ambient space by calculating the change in the normal($\hat{\tau}^a$) to the hypersurface Σ. One cannot freely specify initial data - there are some constraints that the initial data should satisfy. Constraint equations are also found in field theories like electromagnetism where the Lorentz gauge condition has to be imposed on initial data. If the initial data satisfies the constraints, the solutions to the wave equations(with the given initial data) would satisfy the constraints for all times. In GR, the constraint equations also called the Gauss-Codazzi equations are

$$G_{\alpha\beta}\hat{\tau}^\alpha\hat{\tau}^\beta = \frac{1}{2}(R_\Sigma + K^2 - K_{ij}K^{ij}) \quad \text{and} \tag{2.1}$$
$$\frac{\partial\chi^\alpha}{\partial y^i}G_{\alpha\beta}\hat{\tau}^\beta = D_jK_i^j - D_iK. \tag{2.2}$$

$\hat{\tau}^\alpha$ is the future oriented unit normal to Σ, $K = h_{ij}K^{ij}$ and R_Σ is the scalar curvature of Σ. The extrinsic curvature, K_{ij} is given by

$$K_{ij} = \frac{\partial\chi^\alpha}{\partial y^i}\frac{\partial\chi^\beta}{\partial y^j}\nabla_\alpha\hat{\tau}_\beta. \tag{2.3}$$

χ^α refers to coordinate charts on the manifold and $\frac{\partial\chi^\alpha}{\partial y^i}$ denotes the projections. The induced metric, h_{ij} is the pull back of $g_{\mu\nu}$ on Σ. Hence,

$$h_{ij} = \frac{\partial\chi^\alpha}{\partial y^i}\frac{\partial\chi^\beta}{\partial y^j}g_{\alpha\beta}. \tag{2.4}$$

The construction of spacetime from initial data is based on the changes in intrinsic and extrinsic geometry when deformed or evolved under a vector field called the evolution vector field. Evolution is specified through lapse (N) and shift (N^a). The evolution equations given below determine the spacetime close to the hypersurface Σ.

$$\dot{h_{ij}} = -2NK_{ij} + \mathcal{L}_N h_{ij} \;\text{ and} \tag{2.5}$$
$$\dot{K_{ij}} = N\left(R_{ij} - 2K_{il}K^l_j + KK_{ij}\right) - D_iD_jN + \mathcal{L}_N K_{ij}. \tag{2.6}$$

To construct the spacetime close to Σ, coordinates need to be specified on Σ and also in the neighbourhood of Σ. Coordinates for spacetime deformations are uniquely specified by specifying lapse and shift. This construction Lie-drags coordinates between hypersurfaces. The evolution generates a new surface $\Sigma_{t+\Delta t}$ as shown below in Fig.[2.2]. It is important to note that the constraint equations, i.e., Eqs.(2.1) and (2.2) and the evolution equations, i.e., Eqs.(2.5) and (2.6) are a result of projecting the Einstein equations onto Σ_t and perpendicular to Σ_t respectively.

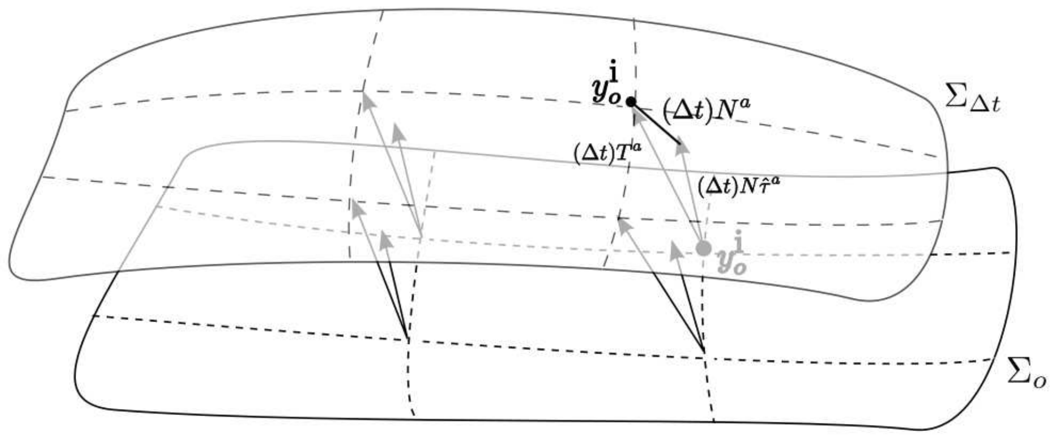


Figure 2.2: Time evolution of surfaces in the 3+1 formalism

Figure based on [1]

2.2 Null hypersurfaces

From null hypersurfaces (like isolated horizons), standard initial data (Σ, h_{ij}, K_{ij}), with the constraint and evolution equations is not enough to fully determine the spacetime in the neighbourhood of Σ. This issue for null hypersurfaces is illustrated in Fig.[2.3]. So one needs to redefine the initial value problem for well-posedness. Basically, this means that we need data that is sufficient to give a non-trivial domain of dependence. This is done by providing data at two intersecting null-surfaces ([24] and [25]). But, this is not the route taken in [1]. Instead, the null surface is broken down as a union of spacelike 2-surfaces and then the usual constraint and evolution equations approach is used. Since there is a difference of two dimensions between the ambient spacetime and the hypersurfaces, there are two extrinsic curvature quantities. This approach is foundationally similar to the initial value formulation and it is found that more on-horizon information is necessary to find the near horizon spacetime.

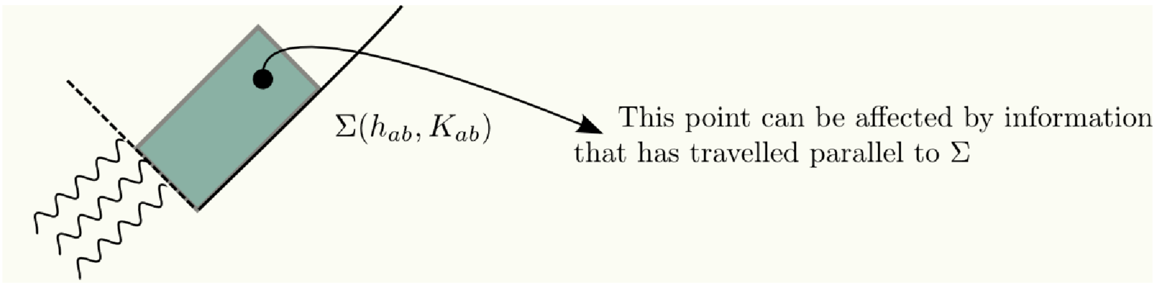


Figure 2.3: Domain of dependence for standard initial data on a null hypersurface

The space normal to the leaves of the horizon(S_v) is spanned by two null vectors, n^a and ℓ^a [1] with inward and outward pointing identifications respectively. These are cross-normalized leaving the normals with a rescaling freedom. The extrinsic curvature analogues $k^{(n)}_{\text{AB}}$ and $k^{(\ell)}_{\text{AB}}$ determine the extrinsic geometry. Taking the n dimensional hypersurface as a union of $n-1$ dimensional foliations, one can define a vector field which is normal to the foliations, tangent to the horizon and evolves the leaves of foliation into each other (like a shift vector). To compute deformations, one appends the horizon with a foliation compatible coordinate system, called Gaussian null coordinates along with an off-horizon coordinate. Now, the initial set of coordinates are Lie-dragged along null geodesics to generate a coordinate system for NHM (Fig.2.4). The deformation quantities can be used to get the corrections to the metric. The near horizon spacetime is obtained in [1] as a Taylor series expansion with the affine parameter as the expansion parameter. The expression

[1] n is used to denote both the dimension of spacetime and a null vector at $\mathcal{H}$. It is assumed that the usage would be unambiguous from the context.

(Eq.87 from [1]) is given below.

$$
\begin{aligned}
ds^2 =& \{-2dvd\rho + 2\mathscr{C}dv^2 + \tilde{q}_{\mathrm{AB}}d\theta^{\mathrm{A}}d\theta^{\mathrm{B}}\} \;+\; \rho\left\{2\kappa_{\mathcal{V}}dv^2 + 4\tilde{\omega}_{\mathrm{A}}dvd\theta^{\mathrm{A}} + 2k^{(n)}_{\mathrm{AB}}d\theta^{\mathrm{A}}d\theta^{\mathrm{B}}\right\} \\
&+ \rho^2 \left\{ \begin{array}{c}
\left(\frac{\tilde{R}}{2} + \tilde{\omega}^{\mathrm{A}}\tilde{\omega}_{\mathrm{A}} - \theta_{(\ell)}\theta_{(n)} + k^{(\ell)}_{\mathrm{AB}}k^{\mathrm{AB}}_{(n)} + \frac{1}{2}Ric_{\alpha\beta}\tilde{q}^{\alpha\beta} + Ric_{\alpha\beta}\ell^{\alpha}n^{\beta}dv^2\right) \\
+\left(2d_{\mathrm{B}}k^{(n)\mathrm{B}}_{\mathrm{A}} - 2d_{\mathrm{A}}\theta_{(n)} - 2\theta_{(n)}\tilde{\omega}_{\mathrm{A}} - 2e^{\alpha}_{\mathrm{A}}Ric_{\alpha\beta}n^{\beta}\right)dvd\theta^{\mathrm{A}} \\
+\left(k^{(n)}_{\mathrm{AB}}k^{(n)C}_{\mathrm{B}} - e^{\alpha}_{A}n^{\beta}e^{\gamma}_{\beta}n^{\delta}C_{\alpha\beta\gamma\delta} - \frac{1}{(n-1)}\tilde{q}_{\mathrm{AB}}Ric_{\gamma\delta}n^{\gamma}n^{\delta}\right)d\theta^{\mathrm{A}}d\theta^{\mathrm{B}}
\end{array} \right\}.
\end{aligned}
\tag{2.7}
$$

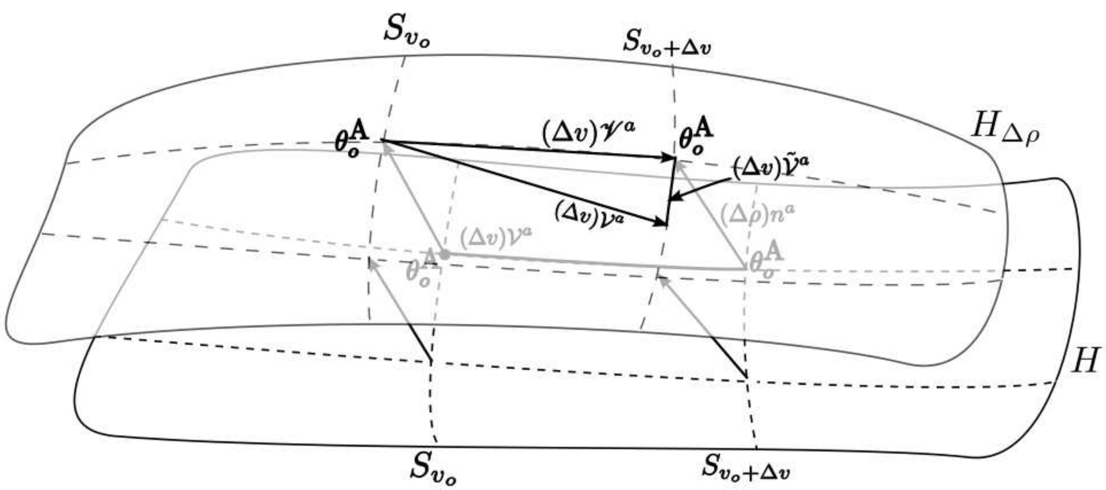


Figure 2.4: Time evolution of surfaces in the 2+1+1 case

Figure based on [1]

In the above expansion, $\mathcal{V} = \ell^a - \mathscr{C}n^a$ is a unique vector field on $\mathcal{H}$ that is normal to the foliations, tangent to the horizon and satisfies $\mathcal{L}_{\mathcal{V}}v = 1$. $\mathscr{C}$ represents a rescaling freedom of the null vectors n^a and ℓ^a. $\mathscr{C} > 0$ means that $\mathcal{H}$ is spacelike, for $\mathscr{C} < 0$, $\mathcal{H}$ is timelike and $\mathscr{C} = 0$ represents a null horizon. $\tilde{q}_{\mathrm{AB}}$ is the induced metric on $\mathcal{H}$. The shift vector field is given by, $\tilde{\mathcal{V}}^{\mathrm{A}} = -\mathcal{L}_{\mathcal{V}}\theta^{\mathrm{A}}$. $\kappa_{\mathcal{V}}$ measures the scaling of null vectors when moving between the leaves of foliations, $\kappa_{\mathcal{V}} = -\mathcal{V}^a n_b \nabla_a \ell^b$. $\kappa_{\mathcal{V}}$ evaluated on an isolated horizon is the surface gravity. $\tilde{\omega}_{\mathrm{A}} = -e^a_{\mathrm{A}}n_b\nabla_a\ell^b$ is the connection on the normal cotangent bundle and it measures the scaling when moving around the leaves. $k^{(X)}_{\mathrm{AB}}$ represents extrinsic curvature analogues along the two directions, $X^a = \ell^a$ and $X^a = n^a$. $\theta_{(X)}$ is the expansion along the direction X^a given by $\theta_{(X)} = \tilde{q}^{\mathrm{AB}} k^{(X)}_{\mathrm{AB}}$. $\tilde{R}$ stands for the 4 dimensional Ricci scalar with d_A representing the derivative on $\mathcal{H}$. $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor. In the expansion, the leading order and sub-leading order terms are completely determined by standard initial data. But for higher orders, more on-horizon information (for instance, the Weyl tensor in second order) is necessary. We provide explicit calculations for the near horizon expansion of WDS in the next section.

2.3 Near horizon spacetime for WDS

As mentioned in Chapter1, WDS is given by Eq.1.36 with Eqs. 1.37 , 1.38 and 1.39 as Einstein equations. As the first step, an ingoing Gaussian null coordinate system is constructed for the horizon. In Eddington-Finkelstein coordinates which are defined by the following transformation of the time coordinate t,

$$t = v - r_*, \tag{2.8}$$

Eq.1.36 becomes

$$\begin{aligned} ds^2 =& -e^{2A}\left(1-\frac{2M}{r}\right)dv^2 + 2e^{B}dvdr + 2e^{2A}C\left(1-\frac{2M}{r}\right)dvd\theta - 2e^{B}Cdrd\theta \\ &+ e^{-2A}r^2\sin^2\theta d\phi^2 + \left\{r^2e^{-2A+2B} - e^{2A}\left(1-\frac{2M}{r}\right)C^2\right\}d\theta^2. \end{aligned} \tag{2.9}$$

$C = C(r,\theta)$ is defined as

$$\begin{aligned} C(r,\theta) &= \frac{\partial r_*}{\partial \theta}, \\ \text{where} \quad r_*(r,\theta) &= \int \frac{e^{-2A+B} dr}{(1 - \frac{2M}{r})}. \end{aligned} \tag{2.10}$$

The horizon ($\mathcal{H}$) remains at $r = r_o = 2M$ ([7] and [9]). It can be checked that $C(r,\theta)$ is well defined on $\mathcal{H}$ (see Appendix C for details). We consider foliations of constant v. The metric at the horizon reads

$$\begin{aligned} ds^2 \triangleq|_{\mathcal{H}} \; & 2e^B dv dr + e^{-2A} r^2 \sin^2\theta d\phi^2 \\ & + \left(r^2 e^{-2A+2B}\right) d\theta^2 - 2e^B C dr d\theta. \end{aligned} \tag{2.11}$$

Here $\triangleq|_{\mathcal{F}}$ means that the expression on the right hand side is evaluated on or at the hypersurface $\mathcal{F}$. The induced metric on the horizon reads

$$dS^2 \triangleq|_{\mathcal{H}} r_o^2 e^{-2A+2B} d\theta^2 + e^{-2A} r_o^2 \sin^2\theta d\phi^2. \tag{2.12}$$

It can be seen that $g_{r\theta}$ is non-vanishing at the horizon. This means that the ingoing future oriented null geodesics are not orthogonal to the foliations. Hence the metric should be rewritten with the appropriate set of geodesics. A suitable pair of cross normalized null normals - ℓ^α and n^α is chosen at the horizon. ℓ^α can be taken to be $\frac{\partial}{\partial v} = (1,0,0,0)$, so, $\ell_\alpha \triangleq|_{\mathcal{H}} (0, e^B, 0, 0)$ and hence, ℓ is null. n^α should be null and tailored to be cross-normalized to ℓ^α. So, if

$$n_\beta \triangleq|_{\mathcal{H}} (D, E, 0, 0), \tag{2.13}$$

$$\text{then,} \quad \frac{D}{E} \triangleq|_{\mathcal{H}} -2\,\frac{{r_o}^2 \mathrm{e}^B}{C^2 \mathrm{e}^{2A}} \quad \text{for } n^\alpha \text{ to be null.} \tag{2.14}$$

From cross normalization,

$$E \triangleq|_{\mathcal{H}} \frac{C^2 \mathrm{e}^{2A}}{2{r_o}^2 \mathrm{e}^{B}} \;\; \text{and} \;\; D = -1. \tag{2.15}$$

To sum up,

$$\ell^\alpha = (1, 0, 0, 0), \tag{2.16}$$

$$\ell_\alpha \triangleq|_{\mathcal{H}} (0, e^B, 0, 0), \tag{2.17}$$

$$n_\alpha \triangleq|_{\mathcal{H}} \begin{bmatrix} -1 & \frac{C^2 \mathrm{e}^{2A}}{2{r_o}^2 \mathrm{e}^{B}} & 0 & 0 \end{bmatrix} \;\; \text{and} \tag{2.18}$$

$$n^\alpha \triangleq|_{\mathcal{H}} \begin{bmatrix} -\frac{C^2 \mathrm{e}^{2A}}{2\mathrm{e}^{2B}{r_o}^2} & -\mathrm{e}^{-B} & -\frac{C\mathrm{e}^{2A}}{\mathrm{e}^{2B}{r_o}^2} & 0 \end{bmatrix}. \tag{2.19}$$

A set of geodesics for which n^α is the tangent vector is identified. These are parametrized by ρ with $\rho = 0$ on $\mathcal{H}$. Geodesics up to ρ^2 are constructed as follows :

$$X^\alpha{}_{(v,\theta,\phi)}(\rho) \approx X^\alpha|_{\rho=0} + \rho \left.\frac{dX^\alpha}{d\rho}\right|_{\rho=0} + \frac{\rho^2}{2} \left.\frac{d^2X^\alpha}{d\rho^2}\right|_{\rho=0} + \frac{\rho^3}{6} \left.\frac{d^3X^\alpha}{d\rho^3}\right|_{\rho=0}, \tag{2.20}$$

where,

$$X^\alpha = [v, r, \theta, \phi]. \tag{2.21}$$

$$X^\alpha|_{\rho=0} = [v, r_0, \theta, \phi]. \tag{2.22}$$

$$\left.\frac{dX^\alpha}{d\rho}\right|_{\rho=0} = n^\alpha \triangleq|_{\mathcal{H}} \begin{bmatrix} -\frac{C^2 \mathrm{e}^{2A}}{2\mathrm{e}^{2B}{r_o}^2}, & -\mathrm{e}^{-B}, & -\frac{C\mathrm{e}^{2A}}{\mathrm{e}^{2B}{r_o}^2}, & 0 \end{bmatrix}. \tag{2.23}$$

$$\left.\frac{d^2X^\alpha}{d\rho^2}\right|_{\rho=0} \triangleq|_{\mathcal{H}} - \Gamma^\alpha_{\beta\gamma} n^\beta n^\gamma. \tag{2.24}$$

$$\left.\frac{d^3X^\alpha}{d\rho^3}\right|_{\rho=0} \triangleq|_{\mathcal{H}} - (\partial_\beta \Gamma^\alpha_{\gamma\delta} + 2\Gamma^\alpha_{\beta\epsilon}\Gamma^\epsilon_{\gamma\delta}) n^\beta n^\gamma n^\delta. \tag{2.25}$$

We have,

$$v = v + \rho v_1 + \frac{\rho^2}{2} v_2 + \frac{\rho^3}{6} v_3, \tag{2.26}$$

$$r = r_o + \rho r_1 + \frac{\rho^2}{2} r_2 + \frac{\rho^3}{6} r_3, \tag{2.27}$$

$$\theta = \theta + \rho\theta_1 + \frac{\rho^2}{2}\theta_2 + \frac{\rho^3}{6}\theta_3, \quad \text{and} \tag{2.28}$$

$$\phi = \phi + \rho\phi_1 + \frac{\rho^2}{2}\phi_2 + \frac{\rho^3}{6}\phi_3. \tag{2.29}$$

The corrections $v_1, v_2, v_3, r_1, r_2, r_3, \theta_1, \theta_2, \theta_3, \phi_1, \phi_2$ and ϕ_3 depend only on θ. Eq.(2.20) can be used to define a coordinate transformation from (v, r, θ, ϕ) to (v, ρ, θ, ϕ) or equivalently from $X^\alpha|_{\rho=0}$ to $X^\alpha(\rho)$. This gives the following second order expansion of the metric.

$$g_{\alpha\beta} dx^\alpha dx^\beta \approx g_{\alpha\beta}(0)\Big|_{\mathcal{H}} dx^\alpha dx^\beta + \rho\, g_{\alpha\beta}(\rho)\Big|_{\mathcal{H}} dx^\alpha dx^\beta + \frac{\rho^2}{2}\, g_{\alpha\beta}(\rho^2)\Big|_{\mathcal{H}} dx^\alpha dx^\beta. \tag{2.30}$$

The first term gives corrections in orders of ρ^0, ρ and ρ^2. The second term gives corrections in ρ and ρ^2 and the third term only gives corrections in ρ^2. The first order corrections to coordinates are

$$v_1 \triangleq|_{\mathcal{H}} -\frac{C^2 \mathrm{e}^{2A}}{2\mathrm{e}^{2B} {r_o}^2}, \tag{2.31}$$

$$r_1 \triangleq|_{\mathcal{H}} - e^{-B}, \tag{2.32}$$

$$\theta_1 \triangleq|_{\mathcal{H}} -\frac{C \mathrm{e}^{2A}}{\mathrm{e}^{2B} {r_o}^2} \quad \text{and} \tag{2.33}$$

$$\phi_1 \triangleq|_{\mathcal{H}} 0. \tag{2.34}$$

The second order corrections are

$$v_2 \triangleq|_{\mathcal{H}} - \frac{Ce^{-6\,B-2\,A}}{8r^5}\left\{\begin{array}{c} C^3\mathrm{e}^{8\,A+B} + 8\,C\mathrm{e}^{4\,A+3\,B}r^2 - 8\,C\mathrm{e}^{4\,A+3\,B}r^3\frac{\partial A}{\partial r} + \\ 8\,C\mathrm{e}^{4\,A+3\,B}r^3\frac{\partial B}{\partial r} + 8\,C^2\mathrm{e}^{6\,A+2\,B}r\frac{\partial B}{\partial \theta} - 8\,C^2\mathrm{e}^{6\,A+2\,B}r\frac{\partial A}{\partial \theta} \\ -8\,\frac{\partial C}{\partial r}\mathrm{e}^{4\,A+3\,B}r^3 - 8\,C\mathrm{e}^{6\,A+2\,B}r\frac{\partial C}{\partial \theta} \end{array}\right\}, \tag{2.35}$$

$$r_2 \triangleq|_{\mathcal{H}} - \frac{\mathrm{e}^{-6\,B}}{2r^3}\left\{ 2\mathrm{e}^{4\,B}r^3\frac{\partial B}{\partial r} + C^2\mathrm{e}^{4\,A+2\,B} + 2\,rC\mathrm{e}^{2\,A+3\,B}\frac{\partial B}{\partial \theta} \right\}, \tag{2.36}$$

$$\theta_2 \triangleq|_{\mathcal{H}} \frac{\mathrm{e}^{-6\,B}}{2r^4}\left\{\begin{array}{c} 3\,\mathrm{e}^{4\,A+2\,B}C^2\frac{\partial B}{\partial \theta} - 2\,\frac{\partial C}{\partial r}\mathrm{e}^{2\,A+3\,B}r^2 + 4\,C\mathrm{e}^{2\,A+3\,B}r \\ -4\,C\mathrm{e}^{2\,A+3\,B}r^2\frac{\partial A}{\partial r} + 4\,C\mathrm{e}^{2\,A+3\,B}r^2\frac{\partial B}{\partial r} - 2\,\mathrm{e}^{4\,A+2\,B}C^2\frac{\partial A}{\partial \theta} \end{array}\right\} \quad \text{and} \tag{2.37}$$

$$\phi_2 = 0. \tag{2.38}$$

With the corrections, we can now construct the NHM. For corrections of order ρ to the coordinates, $X^\alpha(\rho)$, the only contribution for the zeroth order term comes from $\left. g^{(0)}_{\alpha\beta}\right|_H dx^\alpha dx^\beta$. Einstein equations at $\mathcal{H}$ (mentioned in AppendixB) are used to simplify the expressions obtained for the corrections. They are

$$g_{v\rho}(0) = -2, \tag{2.39}$$

$$g_{\theta\theta}(0) \triangleq|_{\mathcal{H}} {r_o}^2\mathrm{e}^{-2A+2B} \quad \text{and} \tag{2.40}$$

$$g_{\phi\phi}(0) \triangleq|_{\mathcal{H}} \mathrm{e}^{-2A}{r_o}^2\sin^2\theta.$$

The zeroth order term is completely determined using standard initial data (which is of course not surprising). For metric corrections in ρ (which will also be referred to as subleading order terms or first order corrections), there would a contribution from both ρ and ρ^2 orders in corrections to the coordinates. The corrections turn out to be

$$g_{vv}(\rho) \triangleq|_{\mathcal{H}} \frac{e^{2A-B}}{r_o}, \tag{2.41}$$

$$g_{v\theta}(\rho) \triangleq|_{\mathcal{H}} \left\{ -2\frac{e^{2A-B}C}{r_o} + 2\frac{\partial B}{\partial \theta} \right\}, \tag{2.42}$$

$$g_{\theta\theta}(\rho) \triangleq|_{\mathcal{H}} \left\{ \begin{array}{c} -2\frac{\partial C}{\partial \theta} - 2r_o e^{-2A+2B} - 2r^2 e^{-2A+B}\frac{\partial A}{\partial r} - 2C\frac{\partial A}{\partial \theta} \\ +e^{2A-B}C^2 - 2r^2 e^{-2A+B}\frac{\partial B}{\partial r} \end{array} \right\} \quad \text{and} \tag{2.43}$$

$$g_{\phi\phi}(\rho) \triangleq|_{\mathcal{H}} \left\{ \begin{array}{c} -2r_o e^{-2A-B}\sin^2\theta + 2r_o^2 e^{-2A-B}\sin^2\theta\frac{\partial A}{\partial r} \\ -2e^{-2B}C\sin\theta\cos\theta - 2e^{-2B}\sin^2\theta C\frac{\partial A}{\partial \theta} \end{array} \right\} \tag{2.44}$$

For second order correction or corrections of order ρ^2, there would be a contribution from all three orders. This part and forward is where more on horizon information is necessary. The corrections obtained are

$$g_{vv}(\rho^2) \triangleq|_{\mathcal{H}} \left\{ \begin{array}{c} \frac{e^{2A-2B}}{r^2} + e^{2A-2B}\frac{\partial^2 A}{\partial \theta^2} + e^{2A-2B}\frac{3\cot\theta}{r^2}\frac{\partial A}{\partial \theta} \\ -C\frac{e^{4A-3B}}{r^3} - \frac{e^{2A-2B}}{r^2}\left(\frac{\partial A}{\partial \theta}\right)^2 + \frac{e^{6A-4BC^2}}{4r^4} \end{array} \right\}, \tag{2.45}$$

$$g_{v\theta}(\rho^2) \triangleq|_{\mathcal{H}} \left\{ \begin{array}{c} \frac{4e^{-B}}{r}\left(\frac{\partial^2 A}{\partial \theta^2}\right)\left(\frac{\partial A}{\partial \theta}\right) + \frac{16e^{-B}\cot\theta}{r}\left(\frac{\partial A}{\partial \theta}\right)^2 \frac{e^{4A-3B}C^2}{r^3}\frac{\partial A}{\partial \theta} \\ -\frac{2Ce^{2A-2B}}{r^2}\left(\frac{\partial A}{\partial \theta}\right)^2 + \frac{2e^{-B}}{r}\frac{\partial^3 A}{\partial \theta^3} + \frac{2\mathrm{cosec}^2\theta e^{-B}}{r}\frac{\partial A}{\partial \theta} \\ -2e^{-B}\cot\theta\frac{\partial^2 A}{\partial \theta^2} - \frac{Ce^{4A-3B}}{r^3}\frac{\partial C}{\partial \theta} - \frac{2e^{2A-2B}}{r^2}\left(\frac{\partial C}{\partial \theta}\right)\left(\frac{\partial A}{\partial \theta}\right) \\ -\frac{4e^{2A-2B}}{r^2} - \frac{2e^{2A-2B}\cot\theta}{r^2}\frac{\partial A}{\partial \theta} + \frac{e^{2A-2B}}{r}\frac{\partial C}{\partial r} \\ +\frac{2e^{2A-2B}C}{r^2} - \frac{C63e^{6A-4B}}{2r^4} + \frac{2Ce^{2A-2B}\cot\theta}{r^2}\frac{\partial A}{\partial \theta} \end{array} \right\}, \tag{2.46}$$

$$g_{\phi\phi}(\rho^2) \triangleq|_{\mathcal{H}} \left\{ \begin{array}{c} 2r^2\sin^2\theta e^{-2A}\left(\frac{\partial A}{\partial r}\right)^2 - \sin^2\theta e^{-2A-2B}\frac{\partial^2 A}{\partial r^2} \\ -4re^{-2A-2B}\frac{\partial A}{\partial r}\sin^2\theta - \frac{2C\sin 2\theta e^{-3B}}{r^2}\frac{\partial A}{\partial r} \\ +\frac{2C\sin 2\theta e^{-3B}}{r} + e^{-2A-2B}\sin^2\theta \end{array} \right\} \quad \text{and} \tag{2.47}$$

$$
g_{\theta\theta}(\rho^2) \triangleq|_{\mathcal{H}} \left\{ \begin{array}{c}
\frac{12C^2e^{2A-2B}}{r^2}\left(\frac{\partial A}{\partial\theta}\right)^2 + \frac{3C^2e^{2A-2B}}{r^2}\frac{\partial^2 A}{\partial\theta^2} - \frac{12Ce^{-B}}{r}\frac{\partial A}{\partial\theta} \\
+\frac{6e^{-B}}{r}\frac{\partial C}{\partial\theta} - 6e^{-B}\left(\frac{\partial C}{\partial\theta}\right)\left(\frac{\partial A}{\partial r}\right) + 12Ce^{-B}\left(\frac{\partial A}{\partial r}\right)\left(\frac{\partial A}{\partial\theta}\right) \\
-3Ce^{-B}\frac{\partial^2 A}{\partial r\partial\theta} + 6e^{-B}\left(\frac{\partial C}{\partial\theta}\right)\left(\frac{\partial B}{\partial r}\right) - 6Ce^{-B}\left(\frac{\partial B}{\partial r}\right)\left(\frac{\partial A}{\partial\theta}\right) \\
+3Ce^{-B}\frac{\partial^2 B}{\partial r\partial\theta} - 10C\frac{\partial A}{\partial\theta} + \frac{e^{2A-2B}}{r^2}\left(\frac{\partial C}{\partial\theta}\right)^2 + \frac{e^{2A-2B}}{2r}\frac{\partial B}{\partial r} \\
+\frac{4e^{4A-3B}C^3}{r^3}\frac{\partial A}{\partial\theta} - \frac{2e^{4A-3B}C^2}{r^3}\frac{\partial C}{\partial\theta} + \frac{8Ce^{2A-2B}}{r^2}\frac{\partial C}{\partial\theta} \\
-e^{2A} - 4re^{-2A} + 3re^{-2A}\frac{\partial B}{\partial r} + 2r^2e^{-2A}\left(\frac{\partial A}{\partial r}\right)^2 \\
-3r^2e^{-2A}\left(\frac{\partial B}{\partial r}\right)\left(\frac{\partial A}{\partial r}\right) - r^2e^{-2A}\frac{\partial^A}{\partial r^2} - \frac{C^2e^{2A-2B}}{2r}\frac{\partial A}{\partial r} \\
+\frac{e^{2A-2B}}{r^2} - e^{-2A}r^2\left(\frac{\partial B}{\partial r}\right)^2 - \frac{C^2e^{2A-2B}}{2r}\frac{\partial B}{\partial r} + \frac{C^2e^{6A-4B}}{2r^4}
\end{array} \right\}. \tag{2.48}
$$

We will review the membrane formalism in the next section. The NHM obtained above will be used to find the stretched horizon for WDS.

2.4 Membrane formalism for WDS

The membrane formalism [2] was strongly motivated by these important results :

1. **Radiation and entropy results - Hawking (1974, 1975, 1976)** ([5], [32] and [33]) :

   Hawking proved that a stationary black hole radiates as though it were a black body with finite surface temperature. With Bekenstein's suggestion, he proved that if the hole is regarded as having an entropy proportional to its surface area, then we can derive the laws of black hole mechanics.

2. **Tidal deformation of the horizon - Hartle and Hawking (1972)** [34] :

   It was proved in this work that external gravitational fields can deform the horizon of a black hole tidally and this deformation is just as it would be if the horizon were a viscous fluid.

3. **Effective charge density for the horizon - Hanni and Ruffini (1973)** [35]:

   They attributed an effective charge density to the horizon of a black hole and showed that when placed in an external static electrical field, the field polarizes the horizon's effective charge distribution.

4. **Horizon's electrical resistivity - Znajek (1976, 1978)** ([36] and [37]) :

   It was shown that when electrical current is run through a black hole, the horizon behaves like it had a surface resistivity of around 30 ohms.

5. **Evolution of the black hole horizon - Independently by Znajek and Damour (1978, 1979, 1982)** ([38], [39] and [40]) :

   All the results (1.-4.) above shed more focus on the horizon from an astrophysicist's point of view. It motivated Damour and Znajek to rewrite the evolution of black hole horizon in a way that one could identify the terms with viscosity, temperature, entropy and electric conductivity.

These identifications and intuitions were still inadequate for astrophysics since the setup was based on a three dimensional null surface (horizon). The generators are null and not timelike. Hence, a distant observer cannot study the physics on the horizon. This setup was then fused with the 3+1 formalism resulting in dynamical equations for a timelike surface very close to the horizon. These equations were approximately the same as that for the true horizon. So physics on this surface could be connected to the external universe. This timelike surface is called the stretched horizon in the literature.

The membrane formalism has a preferred 3+1 split of spacetime close to the true horizon. This split is based on a family of hypersurface orthogonal fiducial observers(FIDO) as illustrated in Fig.[2.5]. Hence there is a choice of time and a preferred slicing. Construction of the stretched horizon means dragging back every point on the event horizon along its null line away from the horizon. We need a spacelike slice with the 3+1 split. The Eddington-Finkelstein coordinates are not suitable since the 3-surfaces are everywhere

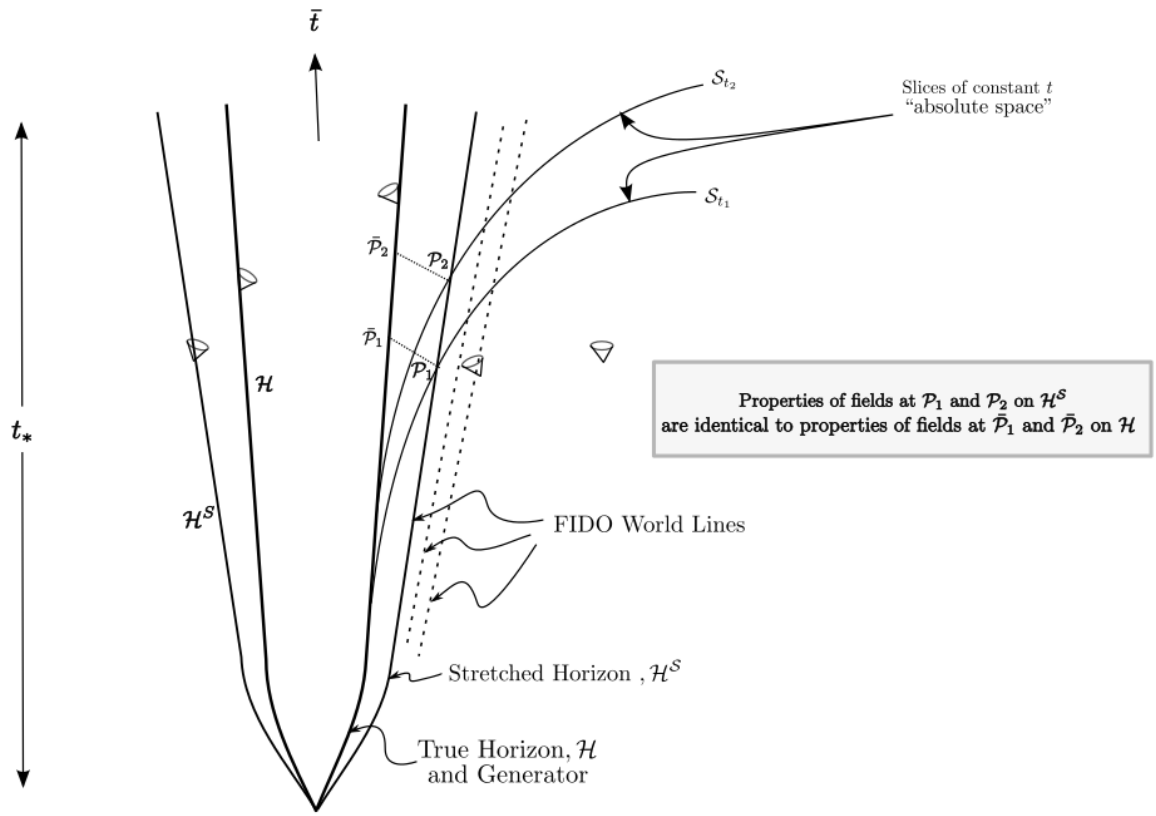


Figure 2.5: FIDO for membrane formalism and 3+1 split
Figure based on [2]

null. The following is a choice of time that works well for WDS.

$$t = v - \frac{1}{2g_H}\ln(2g_H\rho) + \mathcal{O}(\rho). \tag{2.49}$$

Here, g_H is the surface gravity and v is the Eddington Finklestein coordinate. The kinematic properties of the stretched horizon are governed by quantities like shear, expansion and surface gravity. WDS expressed in Eddington-Finkelstein coordinates is Eq.(2.9). The near horizon metric up to order ρ expressed in (v,ρ,θ,ϕ) is :

$$g_{v\rho}(0) \triangleq|_{\mathcal{H}} -2, \tag{2.50}$$

$$g_{\theta\theta}(0) \triangleq|_{\mathcal{H}} {r_o}^2 \mathrm{e}^{-2A+2B}, \tag{2.51}$$

$$g_{\phi\phi}(0) \triangleq|_{\mathcal{H}} \mathrm{e}^{-2A} {r_o}^2 \sin^2\theta, \tag{2.52}$$

$$g_{vv}(\rho) \triangleq|_{\mathcal{H}} \frac{e^{2A-B}}{r_o}, \tag{2.53}$$

$$g_{v\theta}(\rho) \triangleq|_{\mathcal{H}} \left\{-2\frac{e^{2A-B}C}{r_o} + 2\frac{\partial B}{\partial\theta}\right\}, \tag{2.54}$$

$$g_{\theta\theta}(\rho) \triangleq|_{\mathcal{H}} \left\{ \begin{array}{l} -2r_o e^{-2A-B}\sin^2\theta + 2r_o^2 e^{-2A-B}\sin^2\theta\frac{\partial A}{\partial r} \\ -2e^{-2B}C\sin\theta\cos\theta - 2e^{-2B}\sin^2\theta C\frac{\partial A}{\partial\theta} \end{array} \right\} \text{ and} \tag{2.55}$$

$$g_{\phi\phi}(\rho) \triangleq|_{\mathcal{H}} \left\{ \begin{array}{l} 2r_o^2\sin^2\theta\mathrm{e}^{-2A-B}\frac{\partial A}{\partial r} - r_o\sin^2\theta\mathrm{e}^{-2A-B} \\ +C\mathrm{e}^{-2B}\sin^2\theta\frac{\partial A}{\partial\theta} - C\sin\theta\cos\theta\mathrm{e}^{-2B} \end{array} \right\}. \tag{2.56}$$

On changing the time coordinate as per Eq.2.49, we get,

$$g_{t\rho}(0) \triangleq|_{\mathcal{H}} -\left(2 + \frac{1}{g_H\rho}\right), \tag{2.57}$$

$$g_{\rho\rho}(0) \triangleq|_{\mathcal{H}} -\left(\frac{1}{g_H\rho} + \frac{e^{2A}}{4r_o g_H^2\rho}\right), \tag{2.58}$$

$$g_{\theta\theta}(0) \triangleq|_{\mathcal{H}} {r_o}^2\mathrm{e}^{-2A+2B}, \tag{2.59}$$

$$g_{\phi\phi}(0) \triangleq|_{\mathcal{H}} \mathrm{e}^{-2A}{r_o}^2\sin^2\theta, \tag{2.60}$$

$$g_{\rho\theta}(0) \triangleq|_{\mathcal{H}} \frac{1}{2g_H}\left(-2\frac{e^{2A-B}C}{r_o} + 2\frac{\partial B}{\partial\theta}\right), \tag{2.61}$$

$$g_{tt}(\rho) \triangleq|_{\mathcal{H}} \frac{e^{2A-B}}{r_o}, \tag{2.62}$$

$$g_{t\theta}(\rho) \triangleq|_{\mathcal{H}} \frac{1}{2g_H}\left\{ -2\frac{e^{2A-B}C}{r_o} + 2\frac{\partial B}{\partial\theta} \right\}, \tag{2.63}$$

$$g_{\theta\theta}(\rho) \triangleq|_{\mathcal{H}} \left\{ \begin{array}{c} -2r_o e^{-2A-B}\sin^2\theta + 2r_o^2 e^{-2A-B}\sin^2\theta\frac{\partial A}{\partial r} \\ -2e^{-2B}C\sin\theta\cos\theta - 2e^{-2B}\sin^2\theta C\frac{\partial A}{\partial\theta} \end{array} \right\} \text{ and} \tag{2.64}$$

$$g_{\phi\phi}(\rho) \triangleq|_{\mathcal{H}} \left\{ \begin{array}{c} 2r_o^2\sin^2\theta \mathrm{e}^{-2A-B}\frac{\partial A}{\partial r} - r_o\sin^2\theta \mathrm{e}^{-2A-B} \\ +C\mathrm{e}^{-2B}\sin^2\theta\frac{\partial A}{\partial\theta} - C\sin\theta\cos\theta \mathrm{e}^{-2B} \end{array} \right\}. \tag{2.65}$$

We choose our stretched horizon($\mathcal{H}_S$) to be at $\rho = \rho_o$ (with $\rho_o < 0$), which is sufficiently close to and outside the horizon. The induced metric on this surface is :

$$h_{tt} \triangleq|_{\mathcal{H}} \frac{\rho_o e^{2A-B}}{r_o}, \tag{2.66}$$

$$h_{t\theta} \triangleq|_{\mathcal{H}} \rho_o \frac{1}{2g_H} \left\{ -2\frac{e^{2A-B}C}{r_o} + 2\frac{\partial B}{\partial\theta} \right\}, \tag{2.67}$$

$$h_{\theta\theta} \triangleq|_{\mathcal{H}} {r_o}^2\mathrm{e}^{-2A+2B} + \rho_o \left\{ \begin{array}{c} -2r_o e^{-2A-B}\sin^2\theta + 2r_o^2 e^{-2A-B}\sin^2\theta\frac{\partial A}{\partial r} \\ -2e^{-2B}C\sin\theta\cos\theta - 2e^{-2B}\sin^2\theta C\frac{\partial A}{\partial\theta} \end{array} \right\} \text{ and} \tag{2.68}$$

$$h_{\phi\phi} \triangleq|_{\mathcal{H}} \mathrm{e}^{-2A}{r_o}^2\sin^2\theta + \rho_o \left\{ \begin{array}{c} 2r_o^2\sin^2\theta \mathrm{e}^{-2A-B}\frac{\partial A}{\partial r} - r_o\sin^2\theta \mathrm{e}^{-2A-B} \\ +C\mathrm{e}^{-2B}\sin^2\theta\frac{\partial A}{\partial\theta} - C\sin\theta\cos\theta \mathrm{e}^{-2B} \end{array} \right\}. \tag{2.69}$$

This metric represents the stretched horizon for WDS. We will end this chapter with a calculation featuring the quadrupolar distortion of the stretched horizon compared to the standard Schwarzschild case. We will consider expansions of the form [9]:

$$A(r,\theta) = \sum_{i=1}^{\infty} \alpha_i \left(\frac{R}{M}\right)^i P_i, \tag{2.70}$$

$$\text{where} \quad R = R(r,\theta) = \left[\left(1 - \frac{2M}{r}\right)r^2 + M^2\cos^2\theta\right] \text{ and} \tag{2.71}$$

$$P_k = P_k\left(\frac{(r-M)\cos\theta}{R}\right). \tag{2.72}$$

For quadrupolar distortion of A,

$$A = \alpha_2 \left(\frac{R}{m}\right)^2 P_2(R,\theta), \tag{2.73}$$

$$\frac{\partial A}{\partial r} = \alpha_2 \frac{2R}{m}\frac{\partial R}{\partial r} P_2 + \alpha_2 \left(\frac{R}{m}\right)^2 \frac{\partial P_2}{\partial \theta} \quad \text{and} \tag{2.74}$$

$$\frac{\partial A}{\partial \theta} = \alpha_2 \frac{2R}{m}\frac{\partial R}{\partial \theta} P_2 + \alpha_2 \left(\frac{R}{m}\right)^2 \frac{\partial P_2}{\partial \theta}. \tag{2.75}$$

The induced metric h_{ij} at $\mathcal{H}_S$ becomes,

$$\begin{aligned}
h_{tt} &\triangleq|_{\mathcal{H}} \frac{\rho_o e^{2A-B}}{r_o} \\
h_{t\theta} &\triangleq|_{\mathcal{H}}\, \rho_o \frac{1}{2g_H}\left\{-2\frac{e^{2A-B}C}{r_o} + 4\left(\alpha_2 \frac{2R}{m}\frac{\partial R}{\partial \theta}P_2 + \alpha_2\left(\frac{R}{m}\right)^2\frac{\partial P_2}{\partial \theta}\right)\right\}, \\
h_{\theta\theta} &\triangleq|_{\mathcal{H}}\, {r_o}^2 \mathrm{e}^{-2A+2B} + \rho_o \left\{\begin{array}{l} -2r_o e^{-2A-B}\sin^2\theta + 2r_o^2 e^{-2A-B}\sin^2\theta\left(\alpha_2\frac{2R}{m}\frac{\partial R}{\partial r}P_2 + \alpha_2\left(\frac{R}{m}\right)^2\frac{\partial P_2}{\partial \theta}\right) \\ -2e^{-2B}C\sin\theta\cos\theta - 2e^{-2B}\sin^2\theta C\left(\alpha_2\frac{2R}{m}\frac{\partial R}{\partial \theta}P_2 + \alpha_2\left(\frac{R}{m}\right)^2\frac{\partial P_2}{\partial \theta}\right)\end{array}\right\} \quad \text{and} \\
h_{\phi\phi} &\triangleq|_{\mathcal{H}}\, {r_o}^2 \mathrm{e}^{-2A}\sin^2\theta + \rho_o \left\{\begin{array}{l} 2r_o^2\sin^2\theta \mathrm{e}^{-2A-B}\left(\alpha_2\frac{2R}{m}\frac{\partial R}{\partial r}P_2 + \alpha_2\left(\frac{R}{m}\right)^2\frac{\partial P_2}{\partial \theta}\right) - r_o\sin^2\theta\mathrm{e}^{-2A-B} \\ +C\mathrm{e}^{-2B}\sin^2\theta\left(\alpha_2\frac{2R}{m}\frac{\partial R}{\partial \theta}P_2 + \alpha_2\left(\frac{R}{m}\right)^2\frac{\partial P_2}{\partial \theta}\right) - C\sin\theta\cos\theta\mathrm{e}^{-2B}\end{array}\right\}.
\end{aligned} \tag{2.76}$$

Eq.2.76 shows how quadrupolar potentials feature in the WDS stretched horizon. For comparison, the Schwarzschild stretched horizon is given by,

$$h_{tt} \triangleq|_{\mathcal{H}} \frac{\rho_o}{r_o}, \tag{2.77}$$

$$h_{\theta\theta} \triangleq|_{\mathcal{H}} {r_o}^2 - 2\rho_o r_o \sin^2\theta \quad \text{and} \tag{2.78}$$

$$h_{\phi\phi} \triangleq|_{\mathcal{H}} {r_o}^2 \sin^2\theta - \rho_o r_o \sin^2\theta. \tag{2.79}$$

We conclude with a summary of the results obtained in this chapter. We started with the computation of near horizon expansion of WDS using the Taylor series expansion derived in [1]. Using this we defined a time slicing for a 3 + 1 split of spacetime near the horizon which was suitable for the membrane formalism for black holes. We then located a stretched horizon $\mathcal{H}_S$ in close proximity to the true horizon for WDS. We have expressed this stretched horizon for the specific case of quadrupolar distortion and also made a comparison with the Schwarzschild stretched horizon.

Chapter 3

Some near horizon studies with Vaidya spacetime

The Vaidya metric given by Eq.1.28 reads

$$ds^2 = -\left(1 - \frac{2M(v)}{r}\right)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{3.1}$$

As mentioned before it represents a spherically symmetric black hole that is being irradiated by infalling null dust with stress energy tensor

$$T_{ab} = \frac{dM/dv}{4\pi r^2}[dv]_a[dv]_b. \tag{3.2}$$

Here, it is assumed that $\dot{M} = \frac{dM}{dv} << 1$. This means that the solution is perturbatively Schwarzschild. Hence we have the following ansatz as the perturbative expression of hypersurfaces about the Schwarzschild horizon,

$$r^{\mathcal{E}} = 2M(1 + \alpha\dot{M} + \beta\dot{M}^2 + \gamma\ddot{M} + ...). \tag{3.3}$$

Here α, β and γ are successive terms in the perturbation and $\ddot{M} << \dot{M}^2 << \dot{M}$.

In [1], event horizon candidates were obtained for surfaces of the form Eq.(3.3). For Eq.(3.3) to be null, the outgoing null geodesics must satisfy :

$$\frac{dr}{dv} = \frac{1}{2}\left(1 - \frac{2M(v)}{r}\right). \tag{3.4}$$

The event horizon is also a solution to Eq.(3.4). Solutions for null surfaces up to second order are :

$$\alpha = 4, \ \ \beta = 32 \quad \text{and} \quad \gamma = 16M. \tag{3.5}$$

Hence,

$$r^{(1)} = 2M(1 + 4\dot{M}) \tag{3.6}$$

$$r^{(2)} = 2M(1 + 4\dot{M} + 32\dot{M}^2 + 16M\ddot{M} + ...) \tag{3.7}$$

where, $r^{(1)}$ and $r^{(2)}$ indicate the first order and second order null solutions respectively. Hence, they are also called event horizon candidates.

In this chapter, we locate stretched horizons near each of these event horizon candidates (i.e. up to two orders). We finish this chapter describing the relevance of this calculation in defining a larger class of horizons in the near equilibrium regime which would include the event horizon, slowly evolving horizon (see Appendix A for details) and stretched horizon.

3.1 Stretched horizon from slowly evolving horizon(SEH)

First, we show that the induced metric on the stretched horizon is Lorentzian by showing that the determinant is negative. In the same line of reasoning as [1], we consider surfaces

$\mathcal{E}$ of the form:

$$r^{\mathcal{E}} = 2M(1 + \alpha\dot{M} + \beta\dot{M}^2 + \gamma\ddot{M} + ...). \tag{3.8}$$

We reinstate the original assumptions for this expansion i.e, $\dot{M} << 1$ and $\ddot{M} << \dot{M}^2 << \dot{M}$. So we are still perturbatively close to the Schwarzcshild solution. We have, $\rho = -(r - 2M)$, where ρ is the ingoing radial null coordinate (same as defined in Chapter 2). Let ζ be the correction to the null solution [1]. Since the first order null solution is

$$\rho_{(1)} = -8M\dot{M} \quad \text{or} \quad r^{(1)} = 8M\dot{M}, \tag{3.9}$$

it is reasonable to impose and expect

$$\zeta \lesssim r^{(1)} \quad \text{and} \quad \zeta > 0. \tag{3.10}$$

for a stretched horizon at $r(= r_{\mathcal{H}_S}) = r^{(1)} + \zeta$. For the metric to be Lorentzian,

$$\left(1 - \frac{2M}{r}\right) > 2\frac{dr}{dv} \quad \text{which gives} \tag{3.11}$$

$$\zeta > 0. \tag{3.12}$$

$\zeta > 0$ for any timelike surface outside the true horizon. So this result does not say anything new about the stretched horizon or its slowly evolving nature. For a second order correction ζ to the null solution, we get

$$\zeta < \frac{16\dot{M}(3\dot{M} + 1)}{1 - 24\dot{M}} \approx 16\dot{M}. \tag{3.13}$$

This is a meaningful bound which provides a condition on ζ so that the stretched horizon remains small enough to be a perturbative correction.

[1] ζ has been unambiguously used to refer to both the first order and second order corrections.

Now, let's see how the stretched horizon fits into the scheme when we start from a general slowly evolving horizon. First, we consider surfaces of the form

$$\rho^{\mathcal{E}} \approx \rho_{(1)}(v, \theta^A) + \rho_{(2)}(v, \theta^A) + \rho_{(3)}(v, \theta^A) \cdots \tag{3.14}$$

with

$$\begin{aligned} \rho_{(J)} &\sim \mathscr{C}^J, \\ \frac{d\rho_{(J)}}{dv} &\lesssim \mathscr{C}^{J+1} \\ \text{and} \quad \|d_A \rho_{(J)}\| &\lesssim \mathscr{C}^J. \end{aligned} \tag{3.15}$$

Here, $\mathscr{C}$ sets the scale of smallness (see Appendix A for more details). The induced metric would be spacelike. Lets consider the case of near horizon metric up to first order in ρ. The induced metric on $\mathcal{E}$ is

$$\begin{aligned} d\mathcal{E}^2 &= \tilde{q}_{AB} d\theta^A d\theta^B \\ &\quad + \left\{ 2(\mathscr{C} + \rho_{(1)}\mathscr{C}')dv^2 + 2\left(\rho_{(1)}\tilde{q}_{AB}\mathcal{V}'^B - d_A\rho_{(1)}\right) dv d\theta^A + \rho_{(1)}\tilde{q}'_{AB} d\theta^A d\theta^B \right\}. \end{aligned} \tag{3.16}$$

Here, $\mathcal{V}^A$ is the shift vector field, $\kappa_{\mathcal{V}} = \mathcal{V}^a n_b \nabla_a \ell^b$, $\mathcal{V}'^A = 2\tilde{\omega}^A$ with $\tilde{\omega}_A = e^a_A n_b \nabla_a \ell^b$ and $\dot{w} = \frac{dw}{dv}$ for any quantity w. In [1], the same event horizon candidate was also obtained as a null solution to slowly evolving $\rho^{\mathcal{E}}$. We will try to recalculate our stretched horizon result of $\zeta > 0$ for the first order null correction. The determinant of a metric of the form

$$d\Sigma^2 = F dv^2 + 2V_A dv d\theta^A + h_{AB} d\theta^A d\theta^B, \tag{3.17}$$

is given by

$$\det(d\Sigma^2) = (F - h_{AB}V^A V^B) \times \det(h). \tag{3.18}$$

For Eq.(3.16), there is an event horizon candidate for $\rho_{(1)} = -\frac{\mathscr{C}}{\mathscr{C}'} = -\frac{\mathscr{C}}{\kappa_{\mathcal{V}}}$. We need to know if there exists an ϵ such that, for $\rho = \rho_{(1)} + \epsilon$, the determinant given by Eq.(3.18) is negative. We require,

$$(\mathscr{C} + \rho\mathscr{C}') - 2\left(\cancel{\rho^2 q_{AB}\mathcal{V}'^A\mathcal{V}'^B} - 2\rho d_A\rho_{(1)}\mathcal{V}'^A + q^{AB}d_A\rho\ d_B\rho\right) < 0 \tag{3.19}$$

$$\implies \epsilon\mathscr{C}' + 4\rho_{(1)}\mathscr{C}'\mathcal{V}'^A + 4\epsilon\mathscr{C}'\mathcal{V}'^A - 2q^{AB}\mathscr{C}'^2 < 0 \tag{3.20}$$

$$\implies \epsilon < \frac{4\mathscr{C}\mathcal{V}'^A + 2q^{AB}\mathscr{C}'^2}{\mathscr{C}' + 4\mathscr{C}'\mathcal{V}'^A}. \tag{3.21}$$

For Vaidya, this translates to $\epsilon < 0$. From Eq.(3.9), this means that $\zeta > 0$, which is the same as obtained from the perturbative solution.

We conclude this chapter by defining a time slicing for Vaidya to finish the stretched horizon characterisation from the membrane formalism viewpoint for the sake of completion. The lack of a Killing horizon in the Vaidya spacetime makes the definition of surface gravity difficult [41]. But since we are in the near equilibrium regime (slowly evolving), it is reasonable to borrow the time slicing of Schwarzschild for Vaidya spacetime. Also in the slowly evolving regime, it is fair to expect that the surface gravity would not differ much from the equilibrium case. So the time slicing for Vaidya spacetime in the membrane formalism is defined to be

$$t = v - \frac{1}{2g_H}\ln(2g_H\rho) + \mathcal{O}(\rho), \tag{3.22}$$

with g_H being the surface gravity for the Schwarzschild solution.

In this chapter, we have seen that, in the case of Vaidya, one can locate stretched horizons as prescribed by the membrane paradigm starting from a) null solutions obtained perturbatively and b) slowly evolving horizon. This indicates that one may be able to locate stretched horizons near event horizon candidates in the slowly evolving case even in a general setting. Hence, one could define a larger family of horizons including the event horizon, slowly evolving horizon and stretched horizon as its members.

Chapter 4

Junction conditions and thin shells

One other way to look into the physics of Weyl distortions is through the thin shell formalism ([4] and [18]). It was mentioned in Chapter 1 that there are two subspaces of distortions that form the solution space to the Laplace equation. But only solutions singular at infinity are meaningful to be considered as we do not want distortions to be caused by the spacetime singularity at the origin. We hope to replicate the distorting effects of the singularity at infinity by introducing a deforming thin shell of matter in an asymptotically flat spacetime. Achieving this could provide a way of inducing Weyl distortions in any spacetime by just adding the corresponding matter term in the gravitational action. In this chapter, we will review this formalism [4] and then show why it is not straightforward to model Weyl distortions using the same.

The thin shell formalism provides a consistent way of connecting two different spacetimes along a hypersurface. To have a well defined geometry on the hypersurface we need to have the same induced metric from both contributing spacetimes. If the spacetimes we consider are $(\mathcal{M}_1, g^+_{\alpha\beta})$ and $(\mathcal{M}_2, g^-_{\alpha\beta})$, we need the induced metric, h_{ij} to be the same from both sides of a hypersurface Σ along which they are joined. Let $[A]$ represent the jump in quantity A across Σ. Then, we require,

$$[h_{ij}] = 0. \tag{4.1}$$

Failing this, such a cut and paste of spacetimes is not possible at all. Eq.(4.1) is also called the first junction condition. But, satisfying Eq.(4.1) alone does not smoothly join two spacetimes. Even if the induced metric is continuous across the hypersurface, there might be discontinuities at the level of Christoffel symbols. If there are none, then the smoothness of the junction is at least C^2. But if there are discontinuities, then they can be interpreted as a surface stress energy tensor given by,

$$T_{ab} = -\frac{\epsilon}{8\pi}\left([K_{ab}] - [K]h_{ab}\right), \tag{4.2}$$

where K_{ab} is the extrinsic curvature and $\epsilon = \pm 1$ for timelike and spacelike hypersurfaces respectively. Though this interpretation is consistent with the formalism, one needs to check how realistic T_{ab} is. For this, we demand that certain energy conditions[4] are satisfied by T_{ab}. With the assumption that the stress energy tensor admits the following decomposition,

$$T^{\alpha\beta} = \rho\hat{e}_0^\alpha\hat{e}_0^\beta + p_1\hat{e}_1^\alpha\hat{e}_1^\beta + p_2\hat{e}_2^\alpha\hat{e}_2^\beta + p_3\hat{e}_3^\alpha\hat{e}_3^\beta, \tag{4.3}$$

where $\hat{e}_i^\mu (i = 1, 2, 3)$ form an orthonormal basis, the energy conditions can be stated as :

1. Weak energy condition : $\rho \geq 0$ and $\rho + p_i > 0$.

2. Null energy condition : $\rho + p_i \geq 0$.

3. Strong energy condition : $\rho + p_1 + p_2 + p_3 \geq 0$ and $\rho + p_i \geq 0$.

4. Dominant energy condition : $\rho \geq 0$ and $\rho \geq |p_i|$.

We summarise two approaches (two different choices for pairs of spacetimes) we had taken for WDS to outline the difficulties of this approach.

4.1 Weyl interior and Schwarzschild exterior

We begin with the simplest potential solution to obtain an asymptotically flat spacetime. We try to cement the Weyl metric(S_-,Eq.1.15) with the Schwarzschild metric (S_+, Eq.1.21) along the hypersurface Σ defined by $r = r_c$. The induced metrics for Eq.(1.15) and Eq.(1.21) become Eq.(4.4) and Eq.(4.5).

$$d\Sigma^2\Big|_{S_-} \triangleq|_\Sigma - e^{2A}dt^2 + e^{-2A+2B}r_c^2 d\theta^2 + e^{-2A}r_c^2 \sin^2\theta d\phi^2. \tag{4.4}$$

$$d\Sigma^2\Big|_{S_+} \triangleq|_\Sigma - \left(1 - \frac{2M}{r_c}\right)dt^2 + r_c^2 d\theta^2 + r_c^2 \sin^2\theta d\phi^2. \tag{4.5}$$

We know that Eqs. (1.16), (1.17) and (1.18) constitute the Einstein equations for the Weyl metric. As a trial solution for A and B, we can consider separation of variables i.e., $A(r,\theta) = F(r)G(\theta)$. The equation (1.16) decomposes to (4.6) and (4.7) below

$$r^2\frac{d^2F}{dr^2} + 2r\frac{dF}{dr} - \ell(\ell+1)F = 0. \tag{4.6}$$

This is called the Euler differential equation. The solutions are of the form $Ir^\ell + Kr^{-(\ell+1)}$ for different ℓ. The equation for $G(\theta)$ is

$$\frac{d^2G}{d\theta^2} + \cot\theta\frac{dG}{d\theta} + \ell(\ell+1)G = 0. \tag{4.7}$$

This is called the Legendre differential equation and the solutions are Legendre polynomials, i.e., $P_\ell(\cos\theta)$. So,

$$A(r,\theta) = \left(Ir^\ell + Kr^{-(\ell+1)}\right) P_\ell(\cos\theta). \tag{4.8}$$

4.1.1 First junction condition

We need to match the induced metrics on the hypersurface from both the contributing spacetimes. We are considering the possibility of joining two constant r hypersurfaces at Σ. But, we cannot match Eq.(4.4) and Eq.(4.5) at Σ. So this choice of spacetimes and Σ cannot be joined across surfaces of constant r. Below, we will use two different coordinate systems to make it more convenient to work with more general hypersurfaces. We will also allow for asymptotically flat Weyl exteriors.

4.2 WDS inside with asymptotically flat WDS outside

The Weyl metric in cylindrical coordinates is given by Eq.(1.1). As stated before, the Einstein equations reduce to the Laplace equation for A in cylindrical coordinates. So, it is convenient to use cylindrical coordinates because it would make this correspondence more obvious. For this reason, we will try to do all our calculations in cylindrical coordinates. A spacetime is completely determined by specifying $A(\rho, z)$. B is determined from A up to a constant using Eqs.(1.3) - (1.5). A Schwarzschild metric can be fit into the Weyl form using a suitable form for the potentials. In our construction(s), we will try to stitch a Weyl spacetime with another Weyl spacetime which is asymptotically flat. Later, if need be, we can choose a Weyl distorted Schwarzschild spacetime to replace the generic Weyl form in the calculations for more specificity. We assume (1.1) inside a thin shell of matter and the following Weyl metric outside :

$$ds^2 = -\gamma^2 e^{2\tilde{A}} dt^2 + e^{-2\tilde{A}+2\tilde{B}}(d\tilde{\rho}^2 + d\tilde{z}^2) + e^{-2\tilde{A}}\tilde{\rho}^2 d\phi^2. \tag{4.9}$$

The coordinate system used here is $\{t, \tilde{\rho}(\rho, z), \tilde{z}(\rho, z), \phi\}$. This form of the metric guarantees that in this coordinate system too, the Einstein equations that take the form of Eqs.(1.3 - 1.4) for Eq.(1.1) hold for Eq.(4.9). We try to paste the two spacetimes along a hypersurface Σ with coordinates $\{\xi^a = (\tau, \xi, \phi)\}$, where $dt = d\tau$ and ϕ has been borrowed from the indigenous coordinate systems $\{t, \rho, z, \phi\}$ and $\{t, \tilde{\rho}(\rho, z), \tilde{z}(\rho, z), \phi\}$.

4.2.1 First junction condition

The first junction condition requires that the induced metric on Σ be the same when viewed from both sides of the hypersurface. For (1.1) we have,

$$d\Sigma^2(\mathrm{in}) \triangleq|_\Sigma - e^{2A} d\tau^2 + e^{-2A+2B}\left\{\left(\frac{\partial\rho}{\partial\xi}\right)^2 + \left(\frac{\partial z}{\partial\xi}\right)^2\right\} d\xi^2 + \rho(\xi)^2 d\phi^2 \quad \text{and} \tag{4.10}$$

$$d\Sigma^2(\mathrm{out}) \triangleq|_\Sigma - \gamma^2 e^{2\tilde{A}} d\tau^2 + e^{-2\tilde{A}+2\tilde{B}}\left\{\left(\frac{\partial\tilde{\rho}}{\partial\xi}\right)^2 + \left(\frac{\partial\tilde{z}}{\partial\xi}\right)^2\right\} d\xi^2 + \tilde{\rho}(\xi)^2 d\phi^2. \tag{4.11}$$

Here $A, \tilde{A}, B$ and $\tilde{B}$ are functions of only ξ (as are ρ, $\tilde{\rho}$, z and $\tilde{z}$). The first junction condition can be written out as :

$$-e^{2A} = -e^{2\tilde{A}}\gamma^2. \tag{4.12}$$

$$e^{2B-2A}\left\{\left(\frac{\partial\rho}{\partial\xi}\right)^2 + \left(\frac{\partial z}{\partial\xi}\right)^2\right\} = e^{2\tilde{B}-2\tilde{A}}\left\{\left(\frac{\partial\tilde{\rho}}{\partial\xi}\right)^2 + \left(\frac{\partial\tilde{z}}{\partial\xi}\right)^2\right\}. \tag{4.13}$$

$$e^{-2A}\rho^2 = e^{-2\tilde{A}}\tilde{\rho}^2. \tag{4.14}$$

This gives the following :

$$e^{2A} = e^{2\tilde{A}}\gamma^2 \tag{4.15}$$

$$\implies \quad A(\xi) = \tilde{A}(\xi) + \text{ const.} \tag{4.16}$$

$$\text{and} \quad \frac{\partial A}{\partial \xi} = \frac{\partial \tilde{A}}{\partial \xi}. \tag{4.17}$$

$$\implies \quad e^{2B}\gamma^2 \left\{ \left(\frac{\partial \rho}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2 \right\} = e^{2\tilde{B}} \left\{ \left(\frac{\partial \tilde{\rho}}{\partial \xi}\right)^2 + \left(\frac{\partial \tilde{z}}{\partial \xi}\right)^2 \right\} \tag{4.18}$$

$$\text{and} \quad \rho(\xi) = \gamma\tilde{\rho}(\xi). \tag{4.19}$$

Hence, it is possible to satisfy the first junction condition for a hypersurface Σ. This confirms the possibility of a thin shell construction.

4.2.2 Second junction condition

We calculate the jump in the extrinsic curvature across the hypersurface to calculate the stress energy tensor. The tangent vectors to the hypersurface, $e^{\alpha}_{a} = \frac{\partial x^{\alpha}}{\partial y^{a}}$ are given below.
For Σ(in),

$$e^{\alpha}_{\tau} = (1, 0, 0, 0), \tag{4.20}$$

$$e^{\alpha}_{\xi} = (0, \frac{\partial \rho}{\partial \xi}, \frac{\partial z}{\partial \xi}, 0) \tag{4.21}$$

$$\text{and} \quad e^{\alpha}_{\phi} = (0, 0, 0, 1). \tag{4.22}$$

For Σ(out),

$$e^{\alpha}_{\tau} = (1,0,0,0), \tag{4.23}$$

$$e^{\alpha}_{\xi} = (0, \frac{\partial\tilde{\rho}}{\partial\xi}, \frac{\partial\tilde{z}}{\partial\xi}, 0) \tag{4.24}$$

$$\text{and} \quad e^{\alpha}_{\phi} = (0,0,0,1). \tag{4.25}$$

$[e^{\alpha}_{a}] = 0$ and $[n_{\alpha}] = 0$ (jumps across the hypersurface). Hence,

$$\frac{\partial\rho}{\partial\xi} = \frac{\partial\tilde{\rho}}{\partial\xi} \implies \gamma = 1 \text{ and } \frac{\partial\tilde{\rho}}{\partial\rho} = 1.$$
$$\frac{\partial z}{\partial\xi} = \frac{\partial\tilde{z}}{\partial\xi} \implies \frac{\partial\tilde{z}}{\partial z} = 1.$$

From (4.15) and (4.18), this also means,

$$A(\xi) = \tilde{A}(\xi) \quad \text{and} \quad B(\xi) = \tilde{B}(\xi). \tag{4.26}$$

We can now find the normal to the hypersurface with $e^{\alpha}_{a} n_{\alpha} = 0$. Taking

$$n_{\alpha} = [0, F, G, 0], \tag{4.27}$$

we get,

$$F = -G\frac{\partial z}{\partial\xi}\frac{\partial\xi}{\partial\rho}. \tag{4.28}$$

We can now calculate the stress energy tensor by computing the jump in extrinsic curvature across Σ.

$$K_{ab} = n_{\alpha;\beta} e^{\alpha}_{a} e^{\beta}_{b}. \tag{4.29}$$

We have

$$
\begin{aligned}
[K_{\tau\tau}] =& \left[e^{4A-2B} G \left(\frac{\partial A}{\partial \rho} \frac{\partial z}{\partial \xi} \frac{\partial \xi}{\partial \rho} - \frac{\partial A}{\partial z} \right) \right] \\
=& e^{4A-2B} G \left\{ \frac{\partial z}{\partial \xi} \frac{\partial \xi}{\partial \rho} \left[\frac{\partial A}{\partial \rho} \right] - \left[\frac{\partial A}{\partial z} \right] \right\},
\end{aligned} \tag{4.30}
$$

$$
\begin{aligned}
\left[K_{\xi\xi}\right] =& \left[-3G \left(\frac{\partial \rho}{\partial \xi} \frac{\partial z}{\partial \xi} \right) \left(-\frac{\partial A}{\partial \rho} + \frac{\partial B}{\partial \rho} \right) + G \left(\frac{\partial z}{\partial \xi} \right)^2 \left(-\frac{\partial A}{\partial \rho} + \frac{\partial B}{\partial \rho} - 2 \frac{\partial A}{\partial z} + 2 \frac{\partial B}{\partial z} \right) \right] \\
&+ \left[G \left(\frac{\partial z}{\partial \xi} \right)^3 \left(\frac{\partial \xi}{\partial \rho} \right) \left(-\frac{\partial A}{\partial \rho} + \frac{\partial B}{\partial \rho} \right) + G \left(\frac{\partial \rho}{\partial \xi} \right)^2 \left(-\frac{\partial A}{\partial \rho} + \frac{\partial B}{\partial \rho} \right) \right] \\
=& -3G \left(\frac{\partial \rho}{\partial \xi} \frac{\partial z}{\partial \xi} \right) \left[-\frac{\partial A}{\partial \rho} + \frac{\partial B}{\partial \rho} \right] + G \left(\frac{\partial z}{\partial \xi} \right)^2 \left[-\frac{\partial A}{\partial \rho} + \frac{\partial B}{\partial \rho} - 2 \frac{\partial A}{\partial z} + 2 \frac{\partial B}{\partial z} \right] \\
&+ G \left(\frac{\partial z}{\partial \xi} \right)^3 \left(\frac{\partial \xi}{\partial \rho} \right) \left[-\frac{\partial A}{\partial \rho} + \frac{\partial B}{\partial \rho} \right] + G \left(\frac{\partial \rho}{\partial \xi} \right)^2 \left[-\frac{\partial A}{\partial \rho} + \frac{\partial B}{\partial \rho} \right] \quad \text{and}
\end{aligned} \tag{4.31}
$$

$$
\begin{aligned}
\left[K_{\phi\phi}\right] =& \left[G e^{-2B} \rho \left(\frac{\partial z}{\partial \xi} \frac{\partial \xi}{\partial \rho} \frac{\partial A}{\partial \rho} \rho - \frac{\partial A}{\partial z} \rho - 1 \right) \right] \\
=& G e^{-2B} \rho^2 \left\{ \frac{\partial z}{\partial \xi} \frac{\partial \xi}{\partial \rho} \left[\frac{\partial A}{\partial \rho} \right] - \left[\frac{\partial A}{\partial z} \right] \right\}.
\end{aligned} \tag{4.32}
$$

T_{ab} is given below.

$$T_{\tau\tau} = -\frac{1}{8\pi} \left(-[K_{\xi\xi}] h^{\xi\xi} - [K_{\phi\phi}] h^{\phi\phi} \right) h_{\tau\tau}, \tag{4.33}$$

$$T_{\xi\xi} = -\frac{1}{8\pi} \left(-[K_{\tau\tau}] h^{\tau\tau} - [K_{\phi\phi}] h^{\phi\phi} \right) h_{\xi\xi} \quad \text{and} \tag{4.34}$$

$$T_{\phi\phi} = -\frac{1}{8\pi} \left(-[K_{\xi\xi}] h^{\xi\xi} - [K_{\tau\tau}] h^{\tau\tau} \right) h_{\phi\phi}. \tag{4.35}$$

We now need to choose a hypersurface to evaluate the quantities for energy conditions. We consider a sphere $r = \text{const.}$ for the interior. This translates to $\rho = r \sin\theta$ and $z = r\cos\theta$ with $\xi = \theta$. So,

$$\frac{\partial z}{\partial \xi} = \frac{\partial z}{\partial \theta} = -\rho \quad \text{and} \quad \frac{\partial \rho}{\partial \xi} = \frac{\partial \rho}{\partial \theta} = r\cos\theta = z.$$

Quantities needed for checking energy conditions are given below :

$$\begin{aligned}
\rho_0 = T_{\tau\tau}h^{\tau\tau} &= -\frac{1}{8\pi}\left(-[K_{\xi\xi}]h^{\xi\xi} - [K_{\phi\phi}]h^{\phi\phi}\right)h_{\tau\tau}h^{\tau\tau} \\
&= \frac{1}{8\pi}\left([K_{\xi\xi}]h^{\xi\xi} + [K_{\phi\phi}]h^{\phi\phi}\right), && (4.36) \\
p_1 &= \frac{1}{8\pi}\left([K_{\tau\tau}]h^{\tau\tau} + [K_{\phi\phi}]h^{\phi\phi}\right) \quad \text{and} && (4.37) \\
p_2 &= \frac{1}{8\pi}\left([K_{\xi\xi}]h^{\xi\xi} + [K_{\tau\tau}]h^{\tau\tau}\right). && (4.38)
\end{aligned}$$

Where,

$$\begin{aligned}
h^{\xi\xi}[K_{\xi\xi}] =& Ge^{2A-2B}(3\sin\theta\cos\theta + 1 - \sin^2\theta\tan\theta)\left[\rho\left(\frac{\partial A}{\partial\rho}\right)^2 - \rho\left(\frac{\partial A}{\partial z}\right)^2 - \frac{\partial A}{\partial\rho}\right] && (4.39) \\
&+ Ge^{2A-2B}\sin^2\theta\left[4\rho\frac{\partial A}{\partial z}\frac{\partial A}{\partial z} - 2\frac{\partial A}{\partial z}\right], && (4.40) \\
h^{\tau\tau}[K_{\tau\tau}] =& Ge^{2A-2B}\left\{\left[\tan\theta\frac{\partial A}{\partial\rho} - \frac{\partial A}{\partial z}\right]\right\} \quad \text{and} && (4.41) \\
h^{\phi\phi}[K_{\phi\phi}] =& -Ge^{2A-2B}\left\{\left[\tan\theta\frac{\partial A}{\partial\rho} - \frac{\partial A}{\partial z}\right]\right\}. && (4.42)
\end{aligned}$$

Here, $p_1 = 0$ and $\rho = 0$ is a solution which violates the weak energy condition. This could be because of the choice of the hypersurface in our construction. Perhaps a more complex hypersurface would be necessary for such an embedding to obtain non-trivial values. To conclude, we have attempted an embedding of Weyl spacetimes in an asymptotically flat blackground. But, we are yet to obtain a non-trivial solution to this problem. A priori, there is no reason for why this embedding should be impossible. It has also been recently proved that one can achieve this kind of embedding with initial data that is time symmetric [43]. All the cases which we have considered so far were stationary. So, the initial data approach might give better hints. One can be sure about the possibility of such an embedding if we can prove the existence of asymptotically flat initial data with a prescribed metric on the distorted horizon. We leave this as an open problem here.

Chapter 5

Conclusion

5.1 Summary of Results

We summarise our results in this chapter. Broadly, this thesis is concerned with near horizon studies. As examples of sufficient generality, we have used Weyl and Vaidya spacetimes to obtain preliminary results before addressing the same questions in a more general setting.

In Chapter 2, we had reviewed the formalism that was used to construct the near horizon metric using data on the horizon. This data included the derivatives of the metric up to all orders. A Taylor series expansion of the metric is the result of this formalism. We have constructed the near horizon spacetime for WDS and applied the membrane formalism to obtain some physical intuition about the Weyl structure. We have also made a comparison with the stretched horizon for Schwarzschild case.

In Chapter 3, we have located stretched horizons for the Vaidya spacetime and established that one can locate stretched horizons near event horizon candidates for the slowly evolving Vaidya spacetime. We have also verified this result using more general calculations from slowly evolving horizons. To complete the stretched horizon characterisation, we have also defined a time slicing which would work for the case of slowly evolving Vaidya spacetime. We have also discussed how this calculation indicates that a broader class of near equilibrium horizons can be defined.

In Chapter 4, we had reviewed the thin shell formalism which we hoped would give us a matter model for physically explaining Weyl distortions. We have used two approaches here. It was seen that obtaining a reasonable stress energy tensor is difficult. We have also mentioned the initial data approach that could be used to address the same problem. This approach can include the case of non-stationary spacetimes too.

5.2 Future

We are aiming for a completion of the two problems discussed in this thesis. The main direction of study that we are currently pursuing is obtaining laws for what has been named "proxy horizons". These are a universal class of horizons near the equilibrium regime that would include the event horizon, slowly evolving horizon and stretched horizon. In fact, they can be seen as a generalisation of slowly evolving horizons. This was also briefly mentioned in Chapter 3. The inspiration for this work was provided in [1], where it was mentioned as a future aspect of study. This is also similar in spirit to [11].

We are also aiming to prove that Weyl distortions can be modelled in an asymptotically flat setting. The thin shell formalism has been our only approach so far. But, as mentioned towards the end of Chapter 4, the initial data approach can provide more scope by including non-stationary spacetimes for embeddings. We hope to get some results in this direction soon.

Appendices

Appendix A

Notes

A.1 Conical singularities

A conical singularity arises in spacetime when a wedge of $2\pi\delta$ is removed from(or added to) $[0, 2\pi)$ (as shown in Fig.A.1) and the resulting edges are identified. We can see this for the simple case of Minkowski spacetime for which

$$ds^2 = -dt^2 + d\rho^2 + \rho^2 d\phi^2 + dz^2. \tag{A.1}$$

The spacetime is regular and flat everywhere except for $\rho = 0$ where it is singular. This gives a topological defect in the spacetime. In some cases like in Eq.(A.1), it can be cosmetically removed by rescaling. With $\varphi = \alpha\phi$ and $\alpha = 1 - \delta$, Eq.(A.1) takes the form,

$$ds^2 = -dt^2 + d\rho^2 + \alpha^2\rho^2 d\varphi^2 + dz^2. \tag{A.2}$$

One of the examples of spacetimes where this fixing cannot be done is the C-metric which represents two accelerating black holes [14]. Sometimes, addition of fields like in the case of Melvin solutions can provide additional freedom to assist some kind of rescaling as shown in the example above.

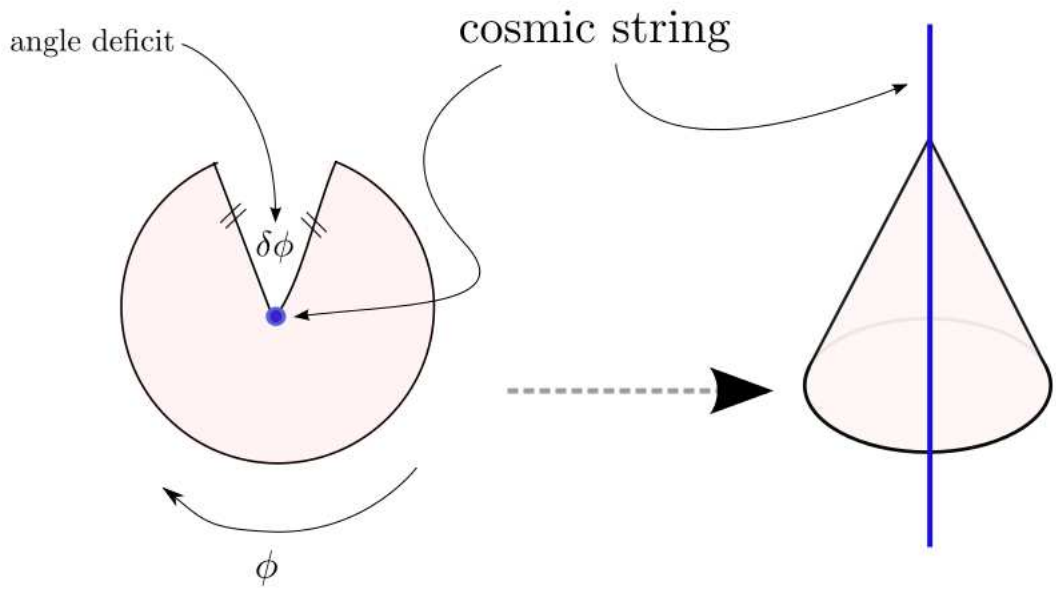


Figure A.1: Conical singularity - Cosmic string

To see the geometric effect of conical singularities, we take two small circles about the axis of rotation (i.e. vanishing ∂_ϕ) at $\theta = 0$ and $\theta = \pi$. Let the range of ϕ be $[0, 2\pi D)$. D is included here to enable some flexibility in rescaling the angular coordinate. Ideally, the ratio of circumference to radius for the two circles should be equal to each other. In case this ratio is not equal to 2π, we can adjust D so that it is finally 2π.

For a circle about $\theta = 0$, we have

$$\frac{\text{Circumference}}{\text{Radius}} = \lim_{\theta \to 0} \frac{\sqrt{g_{\phi\phi}} 2\pi D}{\theta} = \mathcal{R}_1. \tag{A.3}$$

For a circle about $\theta = \pi$, we have

$$\frac{\text{Circumference}}{\text{Radius}} = \lim_{\theta \to \pi} \frac{\sqrt{g_{\phi\phi}} 2\pi D}{\pi - \theta} = \mathcal{R}_2. \tag{A.4}$$

$D(\neq 0)$ cannot be consistently fixed to have $\mathcal{R}_1 = \mathcal{R}_2$. Geometrically the spacetime structure would either have an angle deficit (called cosmic string) or an excess (called cosmic strut) [14]. From Fig.A.1, it can be seen that parallel lines approaching $\rho = 0$ will converge and hence there is a focussing effect on the geodesics. The spacetime is interpreted as caused by an infinite line source with tension. Conical singularities are specifically called cosmic strings (for angle deficit) or cosmic struts (for angle excess).

A.2 Quasilocal horizons

This section is based on [42]. Let us first review the traditional definition of a black hole. An event horizon is defined as the boundary of causal past of future null infinity. This definition is teleological - one has to know the eternal future to locate the black hole region (see Fig.(A.2) below). Let us see how this definition works for the case of an asymptotically flat and spherically symmetric spacetime [5]. Asymptotically flat means that the complement of a compact region is diffeomorphic to a finite union of copies of $\mathbb{R}^4 \setminus \overline{B(0, r)}$ and the metric decays to the flat metric with respect to the radial coordinate on each copy. We then consider the boundary of the conformally compactified flat ends. Future null infinity is denoted by $\mathscr{I}^+$. A black hole region is defined as the boundary of that subset of of $\mathcal{M}$ from which no future-directed null ray reaches $\mathscr{I}^+$. So it is necessary to have the entire spacetime (for conformal compactification) to locate the black hole. So one is forced to ask if there is a more local way to locate the black hole instead of this non-local one. This is where trapped surfaces come in. One can understand trapped surfaces in the following way. Consider a glowing 2-sphere. There would be wavefronts moving in both directions (inward and

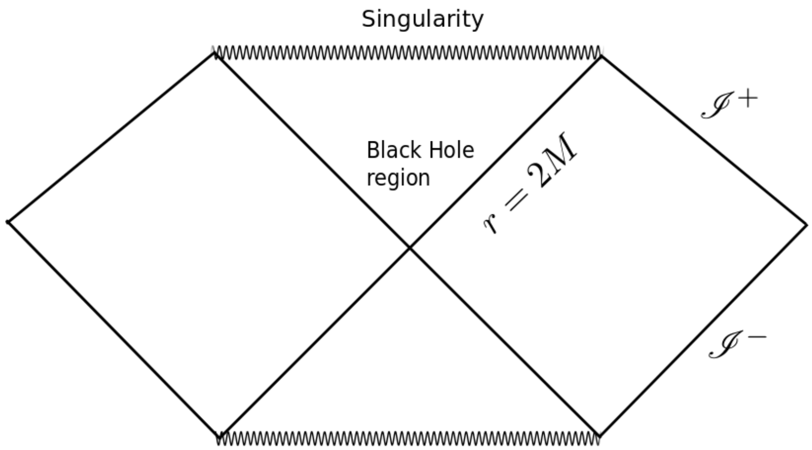


Figure A.2: Schwarzschild singularity

outward). One would expect the area of the inner wavefront to be smaller than 2-sphere and also the outward propagating wavefront. But where there is a trapped surface, gravity is so strong that the both the outward and inward wavefronts have an area less than the glowing 2-sphere that we started out with. Let us see how these are mathematically defined. For this let us substitute the 2-sphere with a spacelike closed 2-surface($\mathscr{S}$) in a 4-dimensional spacetime. This has two orthogonal null directions. Let us denote these by ℓ(outward) and n(inward). These are directions in which null rays travel from $\mathscr{S}$. One can define quantities called expansion scalars θ_ℓ and θ_n which measure the infinitesimal change of area of $\mathscr{S}$ in the respective directions. The expansion in the direction X^a is given by $\theta_{(X)} = \frac{1}{\sqrt{\tilde{q}}}\mathcal{L}_X \sqrt{\tilde{q}}$. A region is called trapped when $\theta_\ell < 0$ and $\theta_n < 0$. When $\theta_n < 0$ and $\theta_\ell = 0$ then $\mathscr{S}$ is called marginally trapped. From here many quasilocal hypersurfaces can be defined for $\mathcal{M}$ (see [42]). A marginally trapped tube is foliated by closed marginally trapped surfaces. Trapped and marginally trapped surfaces always lie inside the black hole(defined traditionally). A dynamical horizon is a marginally trapped tube which is spacelike. An isolated horizon is a marginally trapped tube which is null. An apparent horizon is defined as the boundary of union of all trapped regions in spacelike foliations of a spacetime. Isolated horizons are used to model equilibrium black holes. These definitions are particularly useful in numerical relativity. A short synopsis is given below.

Horizon	θ_ℓ	θ_n
Trapped	< 0	< 0
Marginally trapped	< 0	0
Isolated horizon	$= 0$	< 0
Outer trapped	< 0	$-$
Marginally outer trapped	0	$-$
Weakly trapped	≤ 0	≤ 0

Table A.1: Some quasilocal definitions

A.3 Slowly evolving horizons

The notion of slowly evolving horizons(SEH) was first introduced in [11]. These are understood as nearly isolated horizons. It is defined in the following way [1].

Let $\Delta H = \{\bigcup_v S_v : v_1 \leq v \leq v_2\}$ be a section of future outer trapping horizon with evolution vector field $\mathcal{V} = \ell^a - \mathscr{C} n^a$. ϑ is an evolution parameter defined by,

$$\vartheta^2 / R_H^2 = \text{Max.}\left[\mathscr{C}\left(\|\sigma^{(n)}\|^2 + Ric_{ab} n^a n^b + \theta_{(n)}^2/2\right)\right]. \tag{A.5}$$

R_H is the characteristic length scale for the problem and $\mathscr{C}$ is a smallness parameter. If $\vartheta << 1$ and $\|\mathcal{V}\| = \sqrt{2C} \lesssim \vartheta$, then ΔH is a slowly evolving horizon if on each S_v :

1. the dominant energy condition holds,

2. $|\tilde{\tilde{R}}|, \tilde{\omega}^A\tilde{\omega}^A, |d_A\tilde{\omega}^A|$ and $Ric_{ab}\tilde{q}^{ab} \lesssim 1/R_H^2$,

3. the derivatives of the horizon field tangent to the foliations are at most of the same magnitude as the maximum of the original field. For example, $||d_A\theta_{(n)}|| \lesssim \theta_{(n)}^{max}/R_H$ and

4. derivatives of the horizon field "up" the horizon in the $\mathcal{V}^a$ direction are an order of magnitude ϑ^2/R_H smaller than the original field. For example, $|\mathcal{L}_{\mathcal{V}}\kappa_{\mathcal{V}}| \lesssim (\vartheta^2/R_H)\kappa_{\mathcal{V}}^{max}$ and $|\mathcal{L}_{\mathcal{V}}\mathscr{C}| \lesssim (\vartheta^2/R_H)\mathscr{C}^{max}$.

Here, $X \lesssim Y$ means $X \leq k_o Y$ for some constant k_o. These conditions imply that the geometry of the surface cannot be extreme and that the geometric properties of the horizon change slowly relative to $\mathcal{V}$.

Appendix B

Equations for WDS at $r = 2m$

Here we obtain equations from the Einstein equations on $\mathcal{H}$. This is useful in simplifying expressions obtained for near horizon corrections in Chapter 2 and also while calculating the expansion directly from Eq.(2.7). For WDS, we have

$$\left.\frac{\partial^2 C}{\partial r \partial \theta}\right|_{r=2M} = 0. \tag{B.1}$$

Here,

$$C(r,\theta) = \frac{\partial r_*}{\partial \theta}. \text{ where} \tag{B.2}$$

$$r_*(r,\theta) = \int \frac{e^{-2A+B} dr}{\left(1 - \frac{2M}{r}\right)}. \tag{B.3}$$

$Ric_{\mu\nu} = 0$ at $r = 2M$ gives the following equations :

$$Ric_{vv}|_{r=2M} \equiv 0.$$

$$Ric_{vr}|_{r=2M} = 0 \implies \frac{\partial A}{\partial r} = -\frac{1}{r}\frac{\partial^2 A}{\partial \theta^2} - \frac{\cot\theta}{r}\frac{\partial A}{\partial \theta}.$$

$$Ric_{v\theta}\big|_{r=2M} = 0 \implies \frac{\partial B}{\partial \theta} = 2\frac{\partial A}{\partial \theta}.$$

$$Ric_{v\phi}\Big|_{r=2M} \equiv 0.$$

$$Ric_{rr}\big|_{r=2M} = 0 \implies \left(\frac{\partial^2 A}{\partial r^2}\right) + \frac{2}{r}\left(\frac{\partial A}{\partial r}\right) - \left(\frac{\partial A}{\partial r}\right)^2 - \frac{1}{2}\left(\frac{\partial^2 B}{\partial r^2}\right) = 0.$$

$$Ric_{r\theta}\big|_{r=2M} = 0 \implies \frac{1}{r}\frac{\partial A}{\partial \theta} + \frac{\cot\theta}{2}\frac{\partial B}{\partial r} = \left(\frac{\partial A}{\partial r}\right)\left(\frac{\partial A}{\partial \theta}\right).$$

$$Ric_{\theta\theta}\big|_{r=2M} = 0 \implies \frac{\partial B}{\partial r} = -\frac{2}{r}\left(\frac{\partial A}{\partial \theta}\right)^2 - \frac{2}{r}\frac{\partial^2 A}{\partial \theta^2} + \frac{2\cot\theta}{r}\frac{\partial A}{\partial \theta}.$$

$$Ric_{\phi\phi}\Big|_{r=2M} = 0 \implies \frac{\partial A}{\partial r} = -\frac{1}{r}\frac{\partial^2 A}{\partial \theta^2} - \frac{\cot\theta}{r}\frac{\partial A}{\partial \theta}.$$

Appendix C

Well-definedness of $C(r, \theta)$

We have

$$C(r,\theta) = \frac{\partial r_*}{\partial \theta}, \tag{C.1}$$

$$\text{where} \quad r_*(r,\theta) = \int \frac{e^{-2A+B} dr}{\left(1 - \frac{2M}{r}\right)}. \tag{C.2}$$

The integral is defined for only values other than $r = 2m$ in which case, we have

$$\begin{aligned}
C(r,\theta) = \frac{\partial r_*}{\partial \theta} = \frac{\partial}{\partial \theta} \int \frac{e^{-2A+B} dr}{\left(1 - \frac{2M}{r}\right)} \\
= \int \frac{\partial}{\partial \theta} \left(\frac{e^{-2A+B}}{(1 - \frac{2M}{r})} \right) dr \\
= \int \frac{e^{-2A+B}}{(1 - \frac{2M}{r})} \left(-2\frac{\partial A}{\partial \theta} + \frac{\partial B}{\partial \theta} \right) dr.
\end{aligned}$$

To see if the quantity can be well defined at the horizon, we need to verify that the limit as $r \to 2M$ is finite. In the open interval $(2M, \infty)$,

$$
\begin{aligned}
\frac{dC}{dr} &= \frac{e^{-2A+B}}{(1-\frac{2m}{r})}\left(-2\frac{\partial A}{\partial\theta}+\frac{\partial B}{\partial\theta}\right). \\
\left.\frac{dC}{dr}\right|_{r\to 2M} &= \frac{e^{-2A+B}}{(\frac{2m}{r^2})}\frac{\partial}{\partial r}\left.\left(-2\frac{\partial A}{\partial\theta}+\frac{\partial B}{\partial\theta}\right)\right|_{r=2m} \\
&= \frac{\partial}{\partial\theta}\left.\left(4\cot\theta\frac{\partial A}{\partial\theta}-2\left(\frac{\partial A}{\partial\theta}\right)^2\right)\right|_{r=2m}.
\end{aligned}
$$

Einstein equations have been used to obtain the last expression, which is finite because A and B are at least C^2. This establishes that C is well defined.

Appendix D

Verification of second order corrections

The NHM can be directly computed from Eq.(2.7) which is again mentioned below. The terms in the expression have been computed for WDS and they are given in this section. This calculation would also serve as a verification.

$$ds^2 = \{-2dvd\rho + \cancel{2\mathscr{C}}dv^2 + \tilde{q}_{\mathrm{AB}}d\theta^{\mathrm{A}}d\theta^{\mathrm{B}}\} \;+\; \rho\left\{2\kappa_{\mathcal{V}}dv^2 + 4\tilde{\omega}_{\mathrm{A}}dvd\theta^{\mathrm{A}} + 2k^{(n)}_{\mathrm{AB}}d\theta^{\mathrm{A}}d\theta^{\mathrm{B}}\right\}$$
$$+\rho^2\left\{\begin{array}{c}\left(\frac{\tilde{R}}{2} + \tilde{\omega}^{\mathrm{A}}\tilde{\omega}_{\mathrm{A}} - \cancel{\theta_{(\ell)}\theta_{(n)}} + \cancel{k^{(\ell)}_{\mathrm{AB}}k^{\mathrm{AB}}_{(n)}} + \cancel{\tfrac{1}{2}Ric_{\alpha\beta}\tilde{q}^{\alpha\beta}} + \cancel{Ric_{\alpha\beta}\ell^{\alpha}n^{\beta}}dv^2\right)\\ +\left(2d_{\mathrm{B}}k^{(n)\mathrm{B}}_{\mathrm{A}} - 2d_{\mathrm{A}}\theta_{(n)} - 2\theta_{(n)}\tilde{\omega}_{\mathrm{A}} - \cancel{2e^{\alpha}_{\mathrm{A}}Ric_{\alpha\beta}n^{\beta}}\right)dvd\theta^{\mathrm{A}}\\ +\left(k^{(n)}_{\mathrm{AB}}k^{(n)C}_{\mathrm{B}} - e^{\alpha}_{A}n^{\beta}e^{\gamma}_{\beta}n^{\delta}C_{\alpha\beta\gamma\delta} - \cancel{\tfrac{1}{(n-1)}\tilde{q}_{\mathrm{AB}}Ric_{\gamma\delta}n^{\gamma}n^{\delta}}\right)d\theta^{\mathrm{A}}d\theta^{\mathrm{B}}\end{array}\right\}. \qquad \text{(D.1)}$$

The strikeouts are vanishing terms. The terms in the equation are explained in Chapter 2. The quantities required to compute the non-vanishing terms are give below. For the case of Eq.(1.36),

$$\mathscr{C} = 0 \;\text{ and} \qquad \text{(D.2)}$$

$$\tilde{q}_{\mathrm{AB}}d\theta^{\mathrm{A}}d\theta^{\mathrm{B}} = e^{-2A_o}r_o^2\sin^2\theta d\phi^2 + r_o^2 e^{-2A_o+2B_o}d\theta^2. \qquad \text{(D.3)}$$

$$\text{So, }\; ds^2(0) = -2dvd\rho + e^{-2A_o}r_o^2\sin^2\theta d\phi^2 + r_o^2 e^{-2A_o+2B_o}d\theta^2. \qquad \text{(D.4)}$$

This is the same as Eq.(2.39). Quantities required for the verification of higher order corrections are :

$$\kappa_{\mathcal{V}} = -\mathcal{V}^a n_b \nabla_a \ell^b = -\ell^a n_b \nabla_a \ell^b = \frac{e^{2A-B}}{2r_o}, \tag{D.5}$$

$$\tilde{\omega}_\theta = -e^a_{\mathrm{A}} n_b \nabla_a \ell^b = -\frac{e^{2A-B}C}{2r_o} + \frac{1}{2}\frac{\partial B}{\partial \theta}, \tag{D.6}$$

$$k^{(n)}_{\theta\theta} = \left\{ \begin{array}{c} -r_o e^{-2A-B}\sin^2\theta + r_o^2 e^{-2A-B}\sin^2\theta\frac{\partial A}{\partial r} \\ -e^{-2B}C\sin\theta\cos\theta - e^{-2B}\sin^2\theta C\frac{\partial A}{\partial \theta} \end{array} \right\} \quad \text{and} \tag{D.7}$$

$$k^{(n)}_{\phi\phi} = \left\{ \begin{array}{c} 2r_o e^{-2A}\sin^2\theta - 2r_o^2 e^{-2A}\sin^2\theta\frac{\partial A}{\partial r} \\ +2e^{-2A}r_o^2\sin\theta\cos\theta \end{array} \right\}. \tag{D.8}$$

These values give the same NHM obtained in Chapter 2.

Bibliography

[1] I. Booth, *"Spacetime near isolated and dynamical trapping horizons"*, Phys. Rev. D87 (2013) 024008

[2] R. H. Price and K. S. Thorne, *"Membrane Viewpoint on Black Holes: Properties and Evolution of the Stretched Horizon"*, Phys. Rev. D33 (1986) 915-941

[3] R. M. Wald, *General Relativity*, University of Chicago Press (1984)

[4] E. Poisson, *A Relativist's Toolkit*, Cambridge University Press (2004)

[5] S. W. Hawking and G. F. R. Ellis, *The Large Scale structure of spacetime*, Cambridge Monographs on Mathematical Physics (1973)

[6] H. Weyl, *"On the Theory of Gravitation"*, Ann. Phys. 54, 177 (1917)

[7] R. Geroch and J. B. Hartle, *"Distorted Black Holes"*, J. Math. Phys. 23 (1982) 680

[8] P. Vaidya, *"The Gravitational Field of a Radiating Star"*, Proc. Indian Acad. Sci. Sect. A 33 (1951) 264–276

[9] T. Pilkington, A. Melanson, J. Fitzgerald and I. Booth, *"Trapped and marginally trapped surfaces in Weyl-distorted Schwarzschild solutions"*, Class. Quant. Grav. 28 (2011) 125018

[10] I. Booth and S. Fairhurst, *"Isolated, slowly evolving and dynamical trapping horizons: Geometry and mechanics from surface deformations"*, Phys. Rev. D75 (2004) 084019

[11] I. Booth and S. Fairhurst, *"First Law for slowly evolving horizons"*, Phys. Rev. Lett. 92 (2004) 011102

[12] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Oxford University Press (1992)

[13] S. Fairhurst and B. Krishnan, *"Distorted Black Holes with Charge"*, Int. J. Mod. Phys. D10 (2001) 691-710

[14] J. B. Griffiths and J. Podolsky, *Exact Space-Times in Einstein's General Relativity*, Cambridge University Press (2009)

[15] N. V. Berght and P. Wils, *"The rotation axis for stationary and axisymmetric space-times"*, Class. Quantum Grav. 2 (1985) 229-240

[16] I. Booth, M. Hunt, A. Palomo-Lozano and H. Kunduri, *"Insights from Melvin-Kerr-Newman spacetimes"*, [arXiv:1502.07388]

[17] A. Ashtekar and B. Krishnan, *"Isolated and dynamical horizons and their applications"*, Living. Rev. Rel, 7 (2004) 10

[18] W. Israel, *"Singular hypersurfaces and thin shells in general relativity"*, Nuovo Cim. B44S10 (1966) 1, Nuovo Cim. B48 (1967) 463, Nuovo Cim. B44 (1966) 1

[19] J. Bardeen and G. T. Horowitz, *"The extreme Kerr throat geometry: a vacuum analog of $AdS_2 \times S^2$"*, Phys. Rev. D 60(10) (1999) 104030

[20] A. J. Amsel, G. T. Horowitz, D. Marolf and M. M. Roberts, *"Uniqueness of Extremal Kerr and Kerr–Newman Black Holes"*, Phys. Rev. D 81(2) (2010) 024033.

[21] H. K. Kunduri and J. Lucietti *"A classification of near-horizon geometries of extremal vacuum black holes"*, Journal of Math. Phys., 50 (2009) 082502.

[22] H. K. Kunduri and J. Lucietti, *"Static near-horizon geometries in five dimensions"*, Class. Quant. Grav. 26(24) (2009) 245010

[23] H. K. Kunduri. *"Electrovacuum near-horizon geometries in four and five dimensions"*, Class. Quant. Grav. 28(11) (2011) 114010

[24] H. Friedrich, *"On the regular and the characteristic initial value formulation for Einstein's vacuum field equations"*, Proc. R. Soc. A 375 (1981) 169

[25] B. Krishnan, *"The spacetime in the neighborhood of a general isolated black hole"*, Class. Quant. Grav. 29(20) (2011) 205006

[26] P.T. Chrusciel, G. J. Galloway, D. Pollack, *"Mathematical General Relativity : A Sampler"*, [arxiv.org/1004.1016]

[27] S. Klainerman and F. Nicolo *"On local and global aspects of the Cauchy problem in general relativity"*, Class. Quant. Grav. 16, (1999), R73-R157

[28] L. Anderson, *"The global existence problem in General Relativity"*, [arXiv:gr-qc/9911032]

[29] A. D. Rendall, *"Local and Global Existence Theorems for the Einstein Equations"*, [www.livingreviews.org/Articles/Volume3/2000-1rendall]

[30] Y. Choquet Bruhat and R. Geroch, *" Global Aspects of the Cauchy Problem in General Relativity"*, Comm. Math. Phys. 14 (1969) 329–335

[31] E. Gourgoulhon, *"3+1 Formalism and Bases of Numerical Relativity"*, [arXiv:gr-qc/0703035]

[32] S. W. Hawking, *"Black hole explosions?"*, Nature 248 (5443) (1974) 30–31

[33] S. W. Hawking, *"Particle creation by black holes"*, Comm. Math. Phys. 43, 3 (1975) 199-220

[34] S. W. Hawking and J. B. Hartle, *" Energy and angular momentum flow into a black hole"*, Commun. Math. Phys. 27 (1972) 283

[35] R. Hanni and R. Ruffini, *"Lines of Force of a Point Charge near a Schwarzschild Black Hole"*, Phys. Rev. D8 (1973) 3259

[36] R. Znajek, *"The electric and magnetic conductivity of a Kerr hole "*, Month. Notices of the Royal Astron. Soc. 185 (1978) 833-840

[37] R. Znajek, Ph.D. diss., Univ. of Cambridge (1976)

[38] T. Damour, *"Black-hole eddy currents"*, Phys. Rev. D18 (1978) 3598

[39] T. Damour, Ph.D. diss., University of Paris VI (1979)

[40] T. Damour, *"Surface effects in black hole physics"*, Proceedings of the Second Marcel Grossmann Meeting on General Relativity, ed. R. Ruffini (1982) 587-608

[41] A. B. Nielsen, *"Revisiting Vaidya horizons"*, Galaxies 2 (2014) 62-71

[42] I. Booth, *"Black Hole Boundaries"*, Can. J. Phys. 83 (2005) 1073-1099

[43] C. Mantoulidis and R. Schoen, *"On the Bartnik mass of apparent horizons"*, Class. Quantum Grav. 32 (2015) 205002